On the Production of Dissipation by Interaction of Forced Oscillating Waves in Fluid Dynamics

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Abstract

In the context of some bidimensionnal Navier-Stokes model, we exhibit a family of exact oscillating solutions $\{u_\varepsilon\}_\varepsilon$ defined on some strip $[0,T] \times \mathbb{R}^2$ which does not depend on $\varepsilon \in]0,1]$. The exact solutions is described thanks to a complete expansions which reveal a boundary layer in time $t = 0$. The interactions of the various scales (1, $1/\varepsilon$ and $1/\varepsilon^2$) produce a macroscopic effect given by the addition of a diffusion. To justify the existence of $\{u_\varepsilon\}_\varepsilon$, we need to perform various Sobolev estimates that rely on a refined balance between the informations coming from the hyperbolic and parabolic parts of the equations.

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1 Introduction

In Section 1, we introduce the underlying equations and the functional framework. Then, we state our main result.

1.1 The equations

The time and space variables are $t \in \mathbb{R}_+$ and $x := (x_1, x_2) \in \mathbb{R}^2$. The state variables are the density $\rho \in \mathbb{R}_+$ and the two components u_1 and u_2 of the velocity of the fluid $u := {}^t(u_1, u_2) \in \mathbb{R}^2$. Given a function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, note as usual:

$$\operatorname{div} u := \partial_1 u_1 + \partial_2 u_2 \,, \qquad \partial_1 := \frac{\partial}{\partial x_1} \,, \quad \partial_2 := \frac{\partial}{\partial x_2} \,.$$

In what follows, $\varepsilon \in]0,1]$ is a parameter approaching zero. Introduce the dissipation:

$$\mathcal{P}_\varepsilon u = {}^t \left(\mathcal{P}^1_\varepsilon u \,, \mathcal{P}^2_\varepsilon u \right) := \mu \, \varepsilon^2 \, \Delta_x u + \lambda \, \varepsilon^3 \, \nabla \operatorname{div} \, u$$

where μ, $\lambda \in \mathbb{R}^*_+$ are fixed. Let h be some smooth periodic function with mean zero:

$$h \,:\, \mathbb{T} \longrightarrow \mathbb{R} \,, \qquad \mathbb{T} := \mathbb{R}/\mathbb{Z} \,, \qquad h \in \mathcal{C}^\infty(\mathbb{T};\mathbb{R}) \,, \qquad \int_{\mathbb{T}} h(\theta) \; d\theta = 0 \,.$$

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Consider the following oscillation which is polarized on the second component:

$$F_\varepsilon(x) = {}^t(0, F_\varepsilon^2)(x) := \varepsilon^{-2}\ {}^t\Big(0, \mu\, \partial^2_{\theta\theta} h\left(\varepsilon^{-2} x_1\right)\Big)\,, \qquad \varepsilon \in]0,1]\,.$$

Our starting point is the study of a model based on two-dimensional compressible isentropic equations of the Navier-Stokes type, as can be found in [3, 12], forced by the source term F_ε :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho\, u) = 0\,, \\ \partial_t(\rho u) + \operatorname{div}(\rho\, u \otimes u) + \nabla \rho^\gamma = \rho\,(\mathcal{P}_\varepsilon u - F_\varepsilon)\,, \end{cases} \qquad u \otimes u := \begin{pmatrix} u_1 u_1 & u_1 u_2 \\ u_1 u_2 & u_2 u_2 \end{pmatrix}\,,$$

where γ the adiabatic constant is supposed to be larger than one. To obtain a quasi linear system having a symmetric form, it is classical [13] to introduce the state variable $p := \frac{\sqrt{\gamma}}{C} \rho^C$ with $C := \frac{\gamma - 1}{2}$. Then, we have to deal with:

$$\begin{cases} \partial_t p + u \cdot \nabla p + C\, p \operatorname{div} u = 0\,, \\ \partial_t u + u \cdot \nabla u + C\, p\, \nabla p = \mathcal{P}_\varepsilon u - F_\varepsilon\,. \end{cases} \tag{1}$$

Observe that:

$$\mathcal{P}_\varepsilon\, {}^t(0, h_\varepsilon) - F_\varepsilon = 0\,, \qquad h_\varepsilon(x) := h\Big(\frac{x_1}{\varepsilon^2}\Big)\,, \qquad \forall\, \varepsilon \in]0,1]\,.$$

It follows that, for all $\varepsilon \in]0,1]$, the oscillation ${}^t(0,0,h_\varepsilon)$ satisfies Equation (1).

Our aim is to consider the problem of the *stability* of such families of solutions. To this end, at the initial time $t = 0$, we modify ${}^t(0,0,h_\varepsilon)$ by adding some perturbation. More precisely, we start with:

$$(p, u^1, u^2)(0,x) \;=\; \big(0,0,h\big)\Big(\frac{x_1}{\varepsilon^2}\Big) \;+\; \big(\,\varepsilon^\nu\, q_{0,\varepsilon}\,,\, \varepsilon^M\, v^1_{0,\varepsilon}\,,\, \varepsilon^M\, v^2_{0,\varepsilon}\,\big)\Big(\frac{x_1}{\varepsilon^2}, \frac{x_2}{\varepsilon}\Big) \tag{2}$$

where $(\nu, M) \in \mathbb{N}^2$ with ν large enough and $M \geq 7/2$ (retain that $\nu \gg M$), whereas:

$$(q_{0,\varepsilon}, v^1_{0,\varepsilon}, v^2_{0,\varepsilon})(\theta, y) \in H^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R}^3)\,, \qquad y := \frac{x_2}{\varepsilon} \in \mathbb{R}\,.$$

One effect of the above perturbation is to introduce a dependence on $x_2 \in \mathbb{R}$ (or $y \in \mathbb{R}$). Despite the smallness of ε^ν (and maybe ε^M), when solving (1)-(2), we have to understand the interactions that occur between the very fast oscillations in the direction x_1 (with wavelength ε^2) and the fast variations in the transversal direction x_2 (with wavelength ε). On this way, we are faced with questions about *turbulence*, in the spirit of models proposed in [4, 5, 7].

Another insight on the subject can be obtained by looking at (1) in the variables $(\theta, y) \in \mathbb{T} \times \mathbb{R}$. Then, we are faced with a hyperbolic-parabolic system implying some singular (in $\varepsilon \in]0,1]$) symmetric quasilinear part:

$$\begin{cases} \partial_t p \;\; + \varepsilon^{-2}\, \left(u^1\, \partial_\theta p + \varepsilon\, u^2\, \partial_y p\right) \;\;\; + C\, \varepsilon^{-2}\, p\, \left(\partial_\theta u^1 + \varepsilon\, \partial_y u^2\right) = 0\,, \\ \partial_t u^1 + \varepsilon^{-2}\, \left(u^1\, \partial_\theta u^1 + \varepsilon\, u^2\, \partial_y u^1\right) + C\, \varepsilon^{-2}\, p\, \partial_\theta p = \widetilde{\mathcal{P}}^1_\varepsilon u\,, \\ \partial_t u^2 + \varepsilon^{-2}\, \left(u^1\, \partial_\theta u^2 + \varepsilon\, u^2\, \partial_y u^2\right) + C\, \varepsilon^{-1}\, p\, \partial_y p = \widetilde{\mathcal{P}}^2_\varepsilon u - F^2_\varepsilon\,, \end{cases} \tag{3}$$

and some viscosity which is degenerate on the density and becomes large when $\varepsilon \to 0$:

$$\widetilde{\mathcal{P}}_\varepsilon u := \begin{pmatrix} \widetilde{\mathcal{P}}^1_\varepsilon u \\ \widetilde{\mathcal{P}}^2_\varepsilon u \end{pmatrix} = \frac{1}{\varepsilon^2} \begin{pmatrix} \mu\left(\partial_{\theta\theta} u^1 + \varepsilon^2 \partial_{yy} u^1\right) + \lambda\varepsilon\left(\partial_{\theta\theta} u^1 + \varepsilon \partial_{\theta y} u^2\right) \\ \mu\left(\partial_{\theta\theta} u^2 + \varepsilon^2 \partial_{yy} u^2\right) + \lambda\varepsilon\left(\varepsilon \partial_{\theta y} u^1 + \varepsilon^2 \partial_{yy} u^2\right) \end{pmatrix}.$$

In this article, we show that (for ν large enough and $M \geq 7/2$) the *oscillating Cauchy problem* (1)-(2) is locally well posed in time. We prove (Theorem 1.4) the existence of a time $T \in \mathbb{R}_+^*$ independent of $\varepsilon \in]0,1]$ with solutions $(p_\varepsilon, u_\varepsilon^1, u_\varepsilon^2) = (\varepsilon^\nu\, q_\varepsilon, \varepsilon^M\, v_\varepsilon^1, h_\varepsilon + \varepsilon^M\, v_\varepsilon^2)$ of (1)-(2) on the interval $[0,T]$. We also exhibit (Propositions 1.1 and 1.2) a complete expansion as ε approaches 0 for the expression $(q_\varepsilon, v_\varepsilon^1, v_\varepsilon^2)$. We find (in a sense to be specified later) that $q_\varepsilon \simeq q_\varepsilon^a$ and $v_\varepsilon := (v_\varepsilon^1, v_\varepsilon^2) \simeq v_\varepsilon^a := (v_\varepsilon^{a1}, v_\varepsilon^{a2})$ with:

$$q_\varepsilon^a(t,y,\theta) = \sum_{k=0}^{N+1} \varepsilon^k\, q_k^\varepsilon(t,y,\theta)\,, \qquad v_\varepsilon^a(t,y,\theta) = \sum_{k=0}^{N+1} \varepsilon^k \left(v_k^s(t,y,\theta) + v_k^f\big(\frac{t}{\varepsilon^2}, y, \theta\big)\right). \tag{4}$$

These expansions reveal some time boundary layer at time $t=0$ (recorded at the level of the contribution $v_k^f(\tau,\cdot)$ which is exponentially decreasing with respect to the variable τ) together with some mean evolution behaviour (described by v_k^s). A noticeable aspect is the production of some dissipation when looking at the transport equation (12) on v_k^s. The present approach is not in the continuation of usual $k-\varepsilon$ models [14]. But, in the same spirit, it confirms (and justifies) that the interaction of oscillations can indeed be described at a macroscopic level by the introduction of some *turbulent viscosity*.

1.2 The functional framework

1.2.1 Sobolev spaces

Here K denotes $\mathbb{R}$, $\mathbb{T}\times\mathbb{R}$ or $\mathbb{R}^2$. Let $\alpha = (\alpha_1,\alpha_2) \in \mathbb{N}^2$. The length of α is $|\alpha| := \alpha_1 + \alpha_2$. The notation ∂^α is for the differential operator $\partial_\theta^{\alpha_1}\partial_y^{\alpha_2}$.

- Given $m \in \mathbb{N}\cup\{+\infty\}$ and $p \in \mathbb{N}^*\cup\{+\infty\}$, recall that $W^{m,p}$ is:

$$W^{m,p} := \big\{ f \in L^p(K)\ ;\ \partial^\alpha f \in L^p(K),\ |\alpha| \leq m \big\}\,, \qquad H^m := W^{m,2}\,.$$

When $m \in \mathbb{N}$, the space $W^{m,p}$ can be equipped with the following semi-norm and norm:

$$\forall\, p \in \mathbb{N}^*\cup\{+\infty\}, \qquad \|f\|_{\overset{\circ}{W}{}^{m,p}} := \sum_{\alpha\in\mathbb{N}^2,\ |\alpha|=m} \|\partial^\alpha f\|_{L^p}\ , \qquad \|f\|_{W^{m,p}} := \sum_{k=0}^{m} \|f\|_{\overset{\circ}{W}{}^{k,p}}\ .$$

For $s \in \mathbb{R}_+ \setminus \mathbb{N}$, we can still define spaces $W^{s,p}$ and H^s by interpolation theory.

- Let $(m,n) \in \mathbb{N}^2$ with $n \leq m$. Define the functional spaces:

$$\mathcal{W}_T^{m,n} := \Big\{ f\ ;\ f \in C^j\big([0,T]; W^{m-j,\infty}\big),\ \forall\, j \in \{0,\dots,n\}\Big\}\,,$$
$$\mathcal{H}_T^{m,n} := \Big\{ f\ ;\ f \in C^j\big([0,T]; H^{m-j}\big),\ \forall\, j \in \{0,\dots,n\}\Big\}\,, \qquad T \in \mathbb{R}_+\cup\{+\infty\}\,,$$

which can be seen as Banach spaces when provided with the norms:

$$\|f\|_{\mathcal{W}_T^{m,n}} := \sup_{t\in[0,T]} \sum_{j=0}^{n} \|\partial_t^j f(t,\cdot)\|_{W^{m-j,\infty}}\ , \qquad \|f\|_{\mathcal{H}_T^{m,n}} := \sup_{t\in[0,T]} \sum_{j=0}^{n} \|\partial_t^j f(t,\cdot)\|_{H^{m-j}}\ .$$

- In order to deal with functions $f(t,\cdot)$ defined on $\mathbb{R}_+\times K$, which are exponentially decreasing in the time $t \in \mathbb{R}_+$, and which take their values in the Sobolev space H^s, define:

$$\mathcal{E}_\delta^s := \Big\{ f\ ;\ \sup_{t\in[0,+\infty[} \big(e^{\delta t}\, ||f(t,\cdot)||_{H^s(K)} \big) < +\infty \Big\}\,, \qquad \delta \in \mathbb{R}_+^*\,.$$

- Finally, introduce $\mathcal{E}_\delta^\infty := \bigcap_{j\in\mathbb{N}} \mathcal{E}_\delta^j$, $\mathcal{V}_T^{\infty,0} := \bigcap_{j\in\mathbb{N}} \mathcal{V}_T^{j,0}$ and $\mathcal{V}_T^\infty := \bigcap_{j\in\mathbb{N}} \mathcal{V}_T^{j,j}$ where $\mathcal{V} \in \{\mathcal{H},\mathcal{W}\}$.

1.2.2 Families of functions

In this paragraph, we fix some $\varepsilon_0 \in]0,1]$ and look at families of the type $\{f_\varepsilon\}_{\varepsilon \in]0,\varepsilon_0]}$.

- Assume that $f_\varepsilon \in W^{m,p}(K)$ for all $\varepsilon \in]0,\varepsilon_0]$. To control the size of f_ε, we can use the following weighted anisotropic semi-norm and norm:

$$\forall\, p \in \mathbb{N}^* \cup \{+\infty\}, \qquad \|f_\varepsilon\|_{\overset{\circ}{W}{}^{m,p}_{(1,\varepsilon)}} := \sum_{\alpha \in \mathbb{N}^2,\ |\alpha| = m} \|\varepsilon^{\alpha_1} \partial^\alpha f_\varepsilon\|_{L^p} \ , \qquad \|f_\varepsilon\|_{W^{m,p}_{(1,\varepsilon)}} := \sum_{k=0}^{m} \|f_\varepsilon\|_{\overset{\circ}{W}{}^{k,p}_{(1,\varepsilon)}} \ .$$

We will say that $\{f_\varepsilon\}_\varepsilon$ is bounded in $W^{m,p}_{(1,\varepsilon)}$ when:

$$|\!|\!| f_\cdot |\!|\!|_{W^{m,p}_{(1,\cdot)}} := \sup_{\varepsilon \in]0,\varepsilon_0]} \|f_\varepsilon\|_{W^{m,p}_{(1,\varepsilon)}} < +\infty \, .$$

- Assume that $f_\varepsilon \in \mathcal{V}^{m,n}_T$ for all $\varepsilon \in]0,\varepsilon_0]$ where $\mathcal{V} = \mathcal{W}$ or $\mathcal{V} = \mathcal{H}$. To control the size of f_ε, we can use the following norms:

$$\|f_\varepsilon\|_{\mathcal{V}^{m,n}_{T,\varepsilon}} := \sup_{t \in [0,T]} \sum_{j=0}^{\lfloor n/2 \rfloor} \|\varepsilon^{2j}\, \partial_t^j f_\varepsilon(t,.)\|_{V^{m-j}} \ , \quad \|f_\varepsilon\|_{\mathcal{V}^{m,n}_{T,(1,\varepsilon)}} := \sup_{t \in [0,T]} \sum_{j=0}^{\lfloor n/2 \rfloor} \|\varepsilon^{2j}\, \partial_t^j f_\varepsilon(t,.)\|_{V^{m-j}_{(1,\varepsilon)}} \ .$$

We will say that $\{f_\varepsilon\}_\varepsilon$ is bounded in $\mathcal{V}^{m,n}_{T,\varepsilon}$ or in $\mathcal{V}^{m,n}_{T,(1,\varepsilon)}$ when we have respectively:

$$|\!|\!| f_\cdot |\!|\!|_{\mathcal{V}^{m,n}_{T,\cdot}} := \sup_{\varepsilon \in]0,\varepsilon_0]} \|f_\varepsilon\|_{\mathcal{V}^{m,n}_{T,\varepsilon}} < +\infty \, , \qquad |\!|\!| f_\cdot |\!|\!|_{\mathcal{V}^{m,n}_{T,(1,\cdot)}} := \sup_{\varepsilon \in]0,\varepsilon_0]} \|f_\varepsilon\|_{\mathcal{V}^{m,n}_{T,(1,\varepsilon)}} < +\infty \, .$$

Classical embedding : for $s > 1$ we have $H^s(\mathbb{T} \times \mathbb{R}) \hookrightarrow W^{0,\infty}(\mathbb{T} \times \mathbb{R}) \equiv L^\infty(\mathbb{T} \times \mathbb{R})$. When taking into account the dependence on $\varepsilon \in]0,1]$, there is a loss of powers in ε. Retain here that:

$$\exists\, C \in \mathbb{R}^+_* \, ; \qquad \|f_\varepsilon\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \leq C\ \varepsilon^{-1/2}\ \|f_\varepsilon\|_{H^s_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} \, , \quad \forall\, \varepsilon \in]0,\varepsilon_0] \, . \tag{5}$$

1.2.3 Decomposition of a periodic function

Any function $u \in L^2(\mathbb{T};\mathbb{R})$ can be decomposed as:

$$u(\theta) = \langle u \rangle + u^*(\theta) \, , \qquad \langle u \rangle \in \mathbb{R} \, , \qquad u^* \in L^2(\mathbb{T};\mathbb{R}) \, , \qquad \langle u \rangle \equiv \Pi u := \int_{\mathbb{T}} u(\theta)\ d\theta \, .$$

In what follows, given a symbol $\mathcal{V} \in \{W^{m,p}, H^s, \mathcal{W}^{m,s}_T, \mathcal{H}^s_T, \mathcal{E}^s_\delta\}$, we will manipulate functions $f(t,\theta,y) \in \mathcal{V}(\mathbb{T} \times \mathbb{R})$. We will often decompose f into its mean and oscillating parts according to

$$u(t,\theta,y) = \langle u \rangle (t,y) + u^*(t,\theta,y),$$

with,

$$f^{/\!\!/}(t,y) := \Pi f(t,y) := \langle f(t,\cdot,y) \rangle \in \mathcal{V}^{/\!\!/} := \Pi \mathcal{V}(\mathbb{T} \times \mathbb{R}) \, , \tag{6a}$$

$$f^{\perp}(t,\theta,y) := f^*(t,\theta,y) \in V^{\perp} := (I - \Pi)\mathcal{V}(\mathbb{T} \times \mathbb{R}) \, . \tag{6b}$$

To signal that we consider functions $f(t,\theta,y)$ which do not depend on $\theta \in \mathbb{T}$ ($\Pi f = f$) or whose mean value is zero ($\Pi f = 0$), we will use respectively (as above) the marks $/\!\!/$ and $\perp$. By extension, when dealing with some operator P, we will note

$$P^{/\!\!/} := P\Pi \, , \qquad P^{\perp} := P(I - \Pi) \, . \tag{7}$$

Be careful, in the case of operators, the composition by Π and $I-\Pi$ is put *on the right*.

The derivative ∂_θ acts in the sense of distributions on the space $L^2(\mathbb{T};\mathbb{R})$. We find:

$$\mathcal{K} := \ker \partial_\theta = \{u \equiv c\,;\, c \in \mathbb{R}\}\,, \qquad \mathcal{K}^\perp := (\ker \partial_\theta)^\perp = \big\{u \in L^2(\mathbb{T};\mathbb{R})\,;\, \Pi u = 0\big\}\,.$$

The action ∂_θ has a (right) inverse $\partial_\theta^{-1} : \mathcal{K}^\perp \longrightarrow \mathcal{K}^\perp \cap H^1(\mathbb{T};\mathbb{R})$ which is given by:

$$\partial_\theta^{-1} u(\theta) := \int_0^\theta u(s)\, ds - \int_{\mathbb{T}} \int_0^\theta u(s)\, ds\, d\theta\,, \qquad \forall\, \theta \in \mathbb{T}\,.$$

1.3 Main statements

Since we impose $\nu \gg M \geq 7/2$, the equations on the components u^1 and u^2 can be considered as being partially decoupled from the equation on p. Up to some extent, we can first deal with:

$$\begin{aligned}
\mathcal{L}_1(\varepsilon, q_\varepsilon, v_\varepsilon) &:= \partial_t v_\varepsilon^1 + \varepsilon^{-1}\, h\, \partial_y v_\varepsilon^1 + \varepsilon^{M-2}\, \big(v_\varepsilon^1\, \partial_\theta v_\varepsilon^1 + \varepsilon\, v_\varepsilon^2\, \partial_y v_\varepsilon^1\big) + C\, \varepsilon^{2\nu - M - 2}\, q_\varepsilon\, \partial_\theta q_\varepsilon - \widetilde{\mathcal{P}}_\varepsilon^1 v_\varepsilon\,, \\
\mathcal{L}_2(\varepsilon, q_\varepsilon, v_\varepsilon) &:= \partial_t v_\varepsilon^2 + \varepsilon^{-1}\, h\, \partial_y v_\varepsilon^2 + \varepsilon^{-2}\, \partial_\theta h\, v_\varepsilon^1 + \varepsilon^{M-2}\, \big(v_\varepsilon^1\, \partial_\theta v_\varepsilon^2 + \varepsilon\, v_\varepsilon^2\, \partial_y v_\varepsilon^2\big) \\
&\qquad + C\, \varepsilon^{2\nu - M - 1}\, q_\varepsilon\, \partial_y q_\varepsilon - \widetilde{\mathcal{P}}_\varepsilon^2 v_\varepsilon\,,
\end{aligned}$$

and then look at the remaining part as a transport equation on q_ε :

$$\mathcal{L}_0(\varepsilon, q_\varepsilon, v_\varepsilon) := \partial_t q_\varepsilon + \varepsilon^{-1}\, h\, \partial_y q_\varepsilon + \varepsilon^{M-2}\, \big(v_\varepsilon^1\, \partial_\theta q_\varepsilon + \varepsilon\, v_\varepsilon^2\, \partial_y q_\varepsilon\big) + C\, \varepsilon^{M-2}\, q_\varepsilon\, \big(\partial_\theta v_\varepsilon^1 + \varepsilon\, \partial_y v_\varepsilon^2\big)\,.$$

In what follows, the results will be expressed in terms of the quantities q_ε and v_ε. Of course, the expression

$$(p_\varepsilon, u_\varepsilon^1, u_\varepsilon^2) = (\varepsilon^\nu\, q_\varepsilon, \varepsilon^M\, v_\varepsilon^1, h + \varepsilon^M\, v_\varepsilon^2) \tag{8}$$

is a solution of (3) if and only if:

$$\mathcal{L}_j(\varepsilon, q_\varepsilon, v_\varepsilon) = 0\,, \qquad \forall\, j \in \{0,1,2\}\,. \tag{9}$$

1.3.1 Construction of approximated solutions

We start by constructing approximated solutions for the system (9). The first step is to look at the two last equations of (9), where the $O(\varepsilon^{2\nu-M-2}) \ll 1$ contributions (implying q_ε) are neglected. Thus, we start by considering the system:

$$\begin{cases}
\mathcal{L}_1^a(\varepsilon, v_\varepsilon) := \partial_t v_\varepsilon^1 + \varepsilon^{-1}\, h\, \partial_y v_\varepsilon^1 + \varepsilon^{M-2}\, \big(v_\varepsilon^1\, \partial_\theta v_\varepsilon^1 + \varepsilon\, v_\varepsilon^2\, \partial_y v_\varepsilon^1\big) - \widetilde{\mathcal{P}}_\varepsilon^1 v_\varepsilon\,, \\
\mathcal{L}_2^a(\varepsilon, v_\varepsilon) := \partial_t v_\varepsilon^2 + \varepsilon^{-1}\, h\, \partial_y v_\varepsilon^2 + \varepsilon^{-2}\, \partial_\theta h\, v_\varepsilon^1 + \varepsilon^{M-2}\, \big(v_\varepsilon^1\, \partial_\theta v_\varepsilon^2 + \varepsilon\, v_\varepsilon^2\, \partial_y v_\varepsilon^2\big) - \widetilde{\mathcal{P}}_\varepsilon^2 v_\varepsilon\,.
\end{cases}$$

Proposition 1.1. *Fix an integer $M \in \mathbb{N}$ with $M \geq 2$. Choose any integer $N \in \mathbb{N}$ and any decay rate $\delta \in]0, \mu[$. Select any functions $v_k^0 \in H^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R}^2)$ indexed by $k \in [\![0, N+1]\!]$. There are functions*

$$v_k^s \in \bigcap_{T \in \mathbb{R}_+^*} \mathcal{H}_T^\infty\,, \qquad v_k^f \in \mathcal{E}_\delta^\infty\,, \qquad k \in [\![0, N+1]\!]$$

such that the family $\{v_\varepsilon^a\}_\varepsilon$ defined as indicated in (4) satisfies the following conditions:

i) At the initial time $t = 0$, the trace $v_\varepsilon^a(0,\cdot)$ is prescribed in the following way:

$$v_\varepsilon^a(0, \theta, y) = \sum_{k=0}^{N+1} \varepsilon^k\, v_k^0(\theta, y)\,. \tag{10}$$

ii) For all time $T \in \mathbb{R}_+^*$, for all $m \in \mathbb{N}$, the family $\{\varepsilon^{-N}\, \mathcal{L}_j^a(\varepsilon, v_\varepsilon^a)\}_\varepsilon$ with $j = 1$ or $j = 2$ is bounded in $\mathcal{H}_{T,\varepsilon}^{m,0}$ in the sense that:

$$\sup_{\varepsilon \in]0,1]} \ \sup_{t\in[0,T]} \ \max_{j\in\{1,2\}} \ \big\| \varepsilon^{-N}\, \mathcal{L}_j^a(\varepsilon, v_\varepsilon^a) \big\|_{H^m(\mathbb{T}\times\mathbb{R})} < +\infty \,. \tag{11}$$

Furthermore, the expression Πv_k^s is determined through an equation of the form:

$$\partial_t \Pi v_k^s - \Big(\mu + \frac{1}{\mu} \Pi \left((\partial_\theta^{-1} h)^2 \right) \Big) \partial_{yy} \Pi v_k^s = S_k \tag{12}$$

where the source term S_k depends only on the v_j^s with $j \leq k-1$.

To complete v_ε^a into some approximated solution $(q_\varepsilon^a, v_\varepsilon^a)$ of the complete system (3), there remains to identify the pressure component q_ε^a. To this end, we are satisfied to solve directly the transport equation $\mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a) = 0$ where v_ε^a is adjusted as in Proposition 1.1. Since the expression v_ε^a is a function of the scales of time t and $\frac{t}{\varepsilon^2}$, that goes for $q_\varepsilon^a(t, y, \theta)$ too. In what follows, we will not need to precise the way by which q_ε^a depends on the different time scales $\big(t, \frac{t}{\varepsilon}, \frac{t}{\varepsilon^2}, \cdots\big)$.

Proposition 1.2. *The context is as in Proposition . Note $\{v_\varepsilon^a\}_\varepsilon$ the family issued from the Proposition 1.1. Select functions $q_k^0 \in H^\infty(\mathbb{T}\times\mathbb{R};\mathbb{R}^3)$ indexed by $k \in [\![0, N+1]\!]$. There are functions q_k^ε with:*

$$\{q_k^\varepsilon\}_\varepsilon \in \bigcap_{T\in\mathbb{R}_+^*} \mathcal{H}_T^{\infty,0} \,, \qquad k \in [\![0, N+1]\!] \,, \qquad \varepsilon \in]0,1]$$

such that the expressions q_ε^a defined as indicated in (4) are solutions of the Cauchy problem:

$$\mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a) = 0 \,, \qquad q_\varepsilon^a(0,\theta,y) = \sum_{k=0}^{N+1} \varepsilon^k \ q_k^0(\theta, y) \,. \tag{13}$$

Moreover, for all time $T \in \mathbb{R}_+^*$, for all $m \in \mathbb{N}$ and for all $k \in [\![0, N+1]\!]$, the family $\{q_k^\varepsilon\}_\varepsilon$ is bounded in $\mathcal{H}_{T,(1,\varepsilon)}^{m,0}$ in the sense that:

$$\sup_{\varepsilon\in]0,1]} \ \sup_{t\in[0,T]} \ \|q_k^\varepsilon(t,\cdot)\|_{H^m_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} < +\infty \,. \tag{14}$$

Coming back to $\mathcal{L}_1$ and $\mathcal{L}_2$, we can now make the following statement.

Proposition 1.3. *Select m, M, N, $\nu \in \mathbb{N}$ satisfying:*

$$M \geq 2 \,, \qquad m \geq 2 \qquad \textit{and} \qquad 2\nu - M - 5/2 - (m+1) - N \geq 0. \tag{15}$$

Note $\{v_\varepsilon^a\}_\varepsilon$ and $\{q_\varepsilon^a\}_\varepsilon$ the families obtained with Propositions and . Then, for all $j \in \{1,2\}$, we have:

$$\sup_{\varepsilon\in]0,1]} \ \sup_{t\in[0,T]} \ \big\| \varepsilon^{-N}\, \mathcal{L}_j(\varepsilon, q_\varepsilon^a, v_\varepsilon^a) \big\|_{H^m(\mathbb{T}\times\mathbb{R})} < +\infty \,.$$

1.3.2 Existence and stability result

The parameter $\varepsilon \in]0,1]$ being fixed, the local in time well-posedness of the Cauchy problem (1)-(2) is standard, with corresponding solutions

$$(p_\varepsilon^e, u_\varepsilon^e) = (\varepsilon^\nu\, q_\varepsilon^e, \varepsilon^M\, v_\varepsilon^{e1}, h + \varepsilon^M\, v_\varepsilon^{e2}). \tag{16}$$

It means that, for all $\varepsilon \in]0,1]$, there is a time $T_\varepsilon \in \mathbb{R}_+^*$ (eventually shrinking to zero when ε goes to zero) such that $(q_\varepsilon^e, v_\varepsilon^e)$ with $v_\varepsilon^e := (v_\varepsilon^{e1}, v_\varepsilon^{e2})$ is a solution of (9) on the time interval $[0, T_\varepsilon]$ with initial data as indicated at the level of (10) and (13).

Fix any $R \in \mathbb{N}$. We can always define on the strip $[0, T_\varepsilon]$, two functions q_ε^R and v_ε^R through the identity

$$(q_\varepsilon^e, v_\varepsilon^e) = (q_\varepsilon^a, v_\varepsilon^a) + \varepsilon^R\,(q_\varepsilon^R, v_\varepsilon^R)\,. \tag{17}$$

Two questions are solved below: the existence of exact solutions of (1)-(2) on a time interval $[0, T_c]$ with $T_c \in \mathbb{R}_+^*$ independent of $\varepsilon \in]0,1]$ and the production of controls on $(q_\varepsilon^R, v_\varepsilon^R)$ showing that $(q_\varepsilon^a, v_\varepsilon^a)$ gives indeed some good asymptotic description of $(q_\varepsilon^e, v_\varepsilon^e)$ on $[0, T_c]$.

Theorem 1.4. *[Existence and stability] Assume $\lambda < 4\mu$. Let m, ν, M, N, $R \in \mathbb{N}$ satisfying*

$$M \geq 7/2 \qquad \text{and} \qquad w_m := \min\left(2\nu - M - 5/2 - (m+3) - R, N - R\right) \geq 0\,. \tag{18}$$

Let T_ε the lifespan of the solution of the Cauchy problem (1)-(2). Then there exist $T_c > 0$ and $\varepsilon_{crit} > 0$ such that

$$\forall\, \varepsilon \in]0, \varepsilon_{crit}], \quad T_\varepsilon \geq T_c.$$

Furthermore, the approximated solution ${}^t(q_\varepsilon^a, v_\varepsilon^a)$, constructed thanks to Propositions 1.1 and 1.2, is a relevant expansion for the exact solution ${}^t(q_\varepsilon^e, v_\varepsilon^e)$ (associated with ${}^t(p_\varepsilon^e, u_\varepsilon^e)$ threw Equation (16)) in the sense that the remainder ${}^t(q_\varepsilon^R, v_\varepsilon^R)$, defined in (17), satisfies the following statements.

i) The family $\{q_\varepsilon^R\}_\varepsilon$ is bounded in $\mathcal{H}^{m+3,0}_{T_c,(1,\varepsilon_c)}$:

$$\sup_{\varepsilon \in]0,\varepsilon_c]}\ \sup_{t \in [0,T_c]} \left\| q_\varepsilon^R(t,\cdot) \right\|_{H^{m+3}_{(1,\varepsilon)}} < +\infty. \tag{19}$$

ii) The families $\{v_\varepsilon^{1R}\}_\varepsilon$ and $\{\varepsilon\, v_\varepsilon^{2R}\}_\varepsilon$ are bounded in $\mathcal{H}^{m+3,0}_{T_c,\varepsilon_c}$:

$$\sup_{\varepsilon \in]0,\varepsilon_c]}\ \sup_{t \in [0,T_c]} \left\| v_\varepsilon^{1R}(t,\cdot) \right\|_{H^{m+3}} < +\infty, \qquad \sup_{\varepsilon \in]0,\varepsilon_c]}\ \sup_{t \in [0,T_c]} \left\| \varepsilon\, v_\varepsilon^{2R}(t,\cdot) \right\|_{H^{m+3}} < +\infty. \tag{20}$$

1.4 The context

1.4.1 Historical comments

Let $N \in \mathbb{N}^*$. Consider the following scalar equation of evolution:

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon}\, h(\mathrm{v}) \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon^2}\, \mathcal{Q}\, f_\varepsilon \;=\; S(t,x,\mathrm{v})\,, \qquad f_\varepsilon(t,x,\mathrm{v}) \in \mathbb{R} \tag{21}$$

where $h : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is some smooth function, $\mathcal{Q}$ is some linear operator acting on L^2, and $S(t,x,\mathrm{v})$ is some function depending on the variables $(t,x,\mathrm{v}) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^N$. The unknown is the function $f_\varepsilon(t,x,\mathrm{v})$. Depending on the choice of $\mathcal{Q}$, the Equation (21) can be the neutron equation [2], the Fokker-Planck equation [9] or the Boltzmann transport equation [15]. In this context, it is well-known that the family $\{f_\varepsilon\}_{\varepsilon\in]0,1]}$ has a weak limit , say f_0 as ε goes to 0. In general, the expression f_0 satisfies an equation implying a *drift-diffusion* term of the form $-\operatorname{div}_x(D\,\nabla_x\,\cdot\,)$ where D is some squared matrix depending on the data. The proofs of the related statements rely strongly on the structure of the collision operator $\mathcal{Q}$ which is either a bounded operator or a self-adjoint operator on some weighted version of L^2.

When the operator is less regular or when there is a lack of symmetries [8], the convergence concerns only the mean value ϱ_ε with respect to v, called the density. For some function ϱ_0 satisfying adequate restrictions, we have:

$$\varrho_\varepsilon(t,x) := \int_{\mathbb{R}^N} f_\varepsilon(t,x,\mathrm{v})\, d\mathrm{v} \longrightarrow \varrho_0(t,x)\ . \tag{22}$$

When looking at the structure of $\mathcal{L}^a := {}^t(\mathcal{L}^a_1,\mathcal{L}^a_2)$, there is some analogy with (21). Indeed, the expression $\mathcal{L}^a(\varepsilon,v_\varepsilon)$ can be decomposed into:

$$\mathcal{L}^a(\varepsilon,v_\varepsilon) := \partial_t v_\varepsilon + \frac{1}{\varepsilon}\,\mathcal{T}\,v_\varepsilon + \frac{1}{\varepsilon^2}\,\mathcal{Q}\,v_\varepsilon + \mathcal{LL}(\varepsilon)\,v_\varepsilon + \varepsilon^{M-2}\,\mathcal{NL}(\varepsilon,v_\varepsilon) \tag{23}$$

with:

$$\mathcal{Q} = \begin{pmatrix} -\mu\partial_{\theta\theta} & 0 \\ \partial_\theta h & -\mu\partial_{\theta\theta}\end{pmatrix}, \qquad \mathcal{T} := \begin{pmatrix} h\,\partial_y - \lambda\partial_{\theta\theta} & 0 \\ 0 & h\,\partial_y\end{pmatrix},$$

$$\mathcal{LL}(\varepsilon) := \begin{pmatrix} -\mu\partial_{yy} & -\lambda\partial_{\theta y} \\ -\lambda\partial_{\theta y} & -(\mu+\varepsilon\lambda)\partial_{yy}\end{pmatrix}, \qquad \mathcal{NL}(\varepsilon,v_\varepsilon) := \begin{pmatrix} v^1_\varepsilon\,\partial_\theta v^1_\varepsilon + \varepsilon\, v^2_\varepsilon\,\partial_y v^1_\varepsilon \\ v^1_\varepsilon\,\partial_\theta v^2_\varepsilon + \varepsilon\, v^2_\varepsilon\,\partial_y v^2_\varepsilon\end{pmatrix}.$$

There are many analogies between (21) and (23). In both cases, the hierarchy with respect to the negative powers of ε (namely ε^{-2}, ε^{-1} and ε^0) is the same, with in factor operators sharing analogous structures. Also, the mean value operation $\langle v_\varepsilon\rangle$ is when considering (23) what replaces the integration with respect to v at the level of (22). However, there are two important differences when comparing (21) and (23):

- The Equation (23) is a system ($v_\varepsilon\in\mathbb{R}^2$). When dealing only with the singular part $\varepsilon^{-1}\mathcal{T}+\varepsilon^{-2}\mathcal{Q}$, this problem can be circumvented by first solving the equation on v^1_ε and then by plugging the result into the equation on v^2_ε. However, once the influences of the contributions $\mathcal{LL}$ or $\mathcal{NL}$ are incorporated, such strong decoupling is no more available. When dealing with the full system (23), the discussion must necessarily take into account *vectorial aspects.*

- The operator $\mathcal{Q}$ of (23) is neither selfadjoint nor bounded (on L^2). Up to some extent, it can be viewed as a non selfadjoint perturbation of the selfadjoint action $-\mu\partial_{\theta\theta}\,I$. Still, we can compute the point spectrum $\sigma_P(\mathcal{Q})$ of $\mathcal{Q}: L^2(\mathbb{T};\mathbb{R}^2)\longrightarrow H^{-2}(\mathbb{T};\mathbb{R}^2)$. We find that:

$$\sigma_P(\mathcal{Q}) := \big\{\,\delta\in\mathbb{C}\,;\ \mathcal{Q}-\delta\,I \text{ is not injective}\,\big\} = \big\{\,\mu\,n^2\,;\ n\in\mathbb{N}\,\big\}.$$

From the point of view of central variety theorems, the presence of a (point) spectral gap between the eigenvalue 0 and the other (positive) eigenvalues indicates that there is a separation between two types of behaviours in time, a *slow* one and a *fast* decaying one, for instance in the spirit of [10, 15]. Of course, such a separation is due to the presence of $-\mu\partial_{\theta\theta}\,I$ inside $\mathcal{Q}$. Again, the influence of this dissipation term is what relates (21) and (23).

In other respects, singular systems like (23) have been studied in a purely hyperbolic context, that is when $\mu=\lambda=0$. Then, the discussion is based on tools coming from supercritical nonlinear geometric optics [1, 4, 6].

The asymptotic analysis of (23) under the assumptions retained here is clearly at the interface of what is done in [2, 8, 15] and [1, 4, 6]. Nevertheless, it needs to develop a specific approach which is the matter of the current contribution. In the next paragraph, we give a few indications of our strategy.

1.4.2 Heuristic description

Our analysis of $\mathcal{L}^a$ is based on a discrete Fourier decomposition with respect to $\theta \in \mathbb{T}$. We can expand h as well as $v = {}^t(v^1, v^2)$ into Fourier Series:

$$h = \sum_{k \in \mathbb{Z}^*} h_k \, e^{i\,k\,\theta}\,, \qquad v^j = \sum_{k \in \mathbb{Z}} v_k^j \, e^{ik\theta}\,, \qquad j \in \{1, 2\}\,.$$

Introduce the following linear map:

$$\begin{array}{rcl} \widetilde{\Pi} \,:\, L^2(\mathbb{T};\mathbb{R}^2) & \longrightarrow & L^2(\mathbb{T};\mathbb{R}^2) \\ v = {}^t(v^1, v^2) & \longmapsto & {}^t\Big(v_0^1 \,,\, v_0^2 \,-\, i \displaystyle\sum_{k \in \mathbb{Z}^*} h_k \, \mu^{-1} \, k^{-1} \, v_0^1 \, e^{ik\theta} \Big)\,. \end{array}$$

The application $\widetilde{\Pi}$ is clearly a projector onto the kernel of $\mathcal{Q}$. Retain that:

$$\widetilde{\Pi} \circ \widetilde{\Pi} = \widetilde{\Pi}\,, \qquad \widetilde{\Pi}\, L^2 = \ker \mathcal{Q}\,, \qquad \dim\, (\ker \mathcal{Q}) = 2\,.$$

• **a** • To understand the action of the several operators in $\mathcal{L}^a$ defined in (23), a first approach is to consider the simplified equation:

$$\partial_t \tilde{v}_\varepsilon + \varepsilon^{-2}\, \mathcal{Q}\, \tilde{v}_\varepsilon = 0\,, \qquad \tilde{v}_\varepsilon(0,\theta) = \sum_{k \in \mathbb{Z}} \tilde{v}_k(0)\; e^{ik\theta}\,. \tag{24}$$

The corresponding solution $\tilde{v}_\varepsilon = {}^t(\tilde{v}_\varepsilon^1, \tilde{v}_\varepsilon^2)$ involves components $\tilde{v}_\varepsilon^j$ which can be put in the form

$$\tilde{v}_\varepsilon^j(t) \;=\; \sum_{k \in \mathbb{Z}} \tilde{v}_k^j\big(t, \frac{t}{\varepsilon^2}\big)\; e^{ik\theta}\,, \qquad \tau := \frac{t}{\varepsilon^2}\,, \qquad j \in \{1,2\}\,.$$

For $k = 0$, we find that $\tilde{v}_0^1(t,\tau) = \tilde{v}_0^1(0)$ and:

$$\tilde{v}_0^2(t,\tau) = \tilde{v}_0^2(0) - i \sum_{p \in \mathbb{Z}^*} h_p\, \mu^{-1}\, p^{-1}\; \tilde{v}_{-p}^1(0) + i \sum_{p \in \mathbb{Z}^*} h_p\, \mu^{-1}\, p^{-1}\, \tilde{v}_{-p}^1(0)\; e^{-\mu p^2 \tau}\,.$$

The second (constant) term in $\tilde{v}_0^2(t,\tau)$ is in general non zero and it comes from contributions inside $\tilde{v}_\varepsilon(0,\cdot)$ which are polarized according to $(I - \widetilde{\Pi})\, L^2$. Thus, even if $\tilde{v}_\varepsilon(0,\cdot)$ presses only on the positive point spectrum, the corresponding solution $\tilde{v}_\varepsilon$ is not necessarily exponentially decreasing in time. We can see here a first effect of the nonselfadjoint part inside $\mathcal{Q}$.

For $k \in \mathbb{Z}^*$, noting $\aleph := \{p \in \mathbb{Z}^*; p \neq k, p \neq 2k\}$, we have $\tilde{v}_k^1(t,\tau) = \tilde{v}_k^1(0)\, e^{-\mu k^2 \tau}$ and:

$$\begin{aligned} \tilde{v}_k^2(t,\tau) = & - i\, h_k\, \mu^{-1}\, k^{-1}\; \tilde{v}_0^1(0) - 2\, i\, k\, h_{2k}\;\, \tilde{v}_{-k}^1(0)\;\, \tau\;\, e^{-\mu k^2 \tau} \\ & + i \sum_{p \in \aleph} h_p\, \mu^{-1}\, (p - 2k)^{-1}\;\, \tilde{v}_{k-p}^1(0)\;\, e^{-\mu (k-p)^2 \tau} - i \sum_{p \in \aleph} h_p\, \mu^{-1}\, (p-2k)^{-1}\;\, \tilde{v}_{k-p}^1(0)\;\, e^{-\mu k^2 \tau} \\ & + \big[\tilde{v}_k^2(0) + i\, h_k\, \mu^{-1}\, k^{-1}\;\, \tilde{v}_0^1(0) \big]\;\, e^{-\mu k^2 \tau}\,. \end{aligned}$$

By bringing together all constant terms (in τ) inside an expression $v_k^s(t,\theta)$ which here does not depend on t, these formulas fit with a decomposition like (4).

• **b** • Next, consider the more elaborated model:

$$\partial_t \check{v}_\varepsilon + \varepsilon^{-1}\, \mathcal{T}\, \check{v}_\varepsilon + \varepsilon^{-2}\, \mathcal{Q}\, \check{v}_\varepsilon = 0\,, \qquad \check{v}_\varepsilon(0,y,\theta) = \sum_{k \in \mathbb{Z}} \check{v}_k(0,y)\; e^{ik\theta}\,. \tag{25}$$

One can expect that the intermediate singular term $\varepsilon^{-1}\, \mathcal{T}$ produces the scaling t/ε. However, such an effect does not appear here. On the one hand, the contributions polarized according to $(I - \widetilde{\Pi})\, L^2$ are mainly handled as in paragraph **a**. On the other hand, the $\widetilde{\Pi}\, L^2$ parts disappear by a combination of two arguments:

- Due to the relation $\int_{\mathbb{T}} hh' d\theta = 0$, we can use the following algebraic identity:

$$\widetilde{\Pi} \circ \mathcal{T} \circ \widetilde{\Pi} \equiv 0 \,. \tag{26}$$

- We can absorb the extra term $(I - \widetilde{\Pi})\mathcal{T}\widetilde{\Pi}$ through some ellipticity inside $\mathcal{Q}$. Indeed, in what follows, we seek $\check{v}_\varepsilon$ as an expansion of the form $\check{v}_\varepsilon = \check{v}_0 + \varepsilon\, \check{v}_1 + O(\varepsilon^2)$. Assuming that $\check{v}_0 = \widetilde{\Pi}\check{v}_0 \in \ker \mathcal{Q}$, we can observe that:

$$(I - \widetilde{\Pi})\,(\varepsilon^{-1}\,\mathcal{T} + \varepsilon^{-2}\,\mathcal{Q})\,(\check{v}_0 + \varepsilon\,\check{v}_1) \;=\; \varepsilon^{-1}\;\big[(I - \widetilde{\Pi})\mathcal{T}\widetilde{\Pi}\,\check{v}_0 + (I - \widetilde{\Pi})\mathcal{Q}(I - \widetilde{\Pi})\,\check{v}_1\big] + O(1)\,.$$

  Now, the idea is to adjust $\check{v}_1$ conveniently in order to remove the $O(\varepsilon^{-1})$ contribution.

In practice, the implementation of these arguments must be done with care because the different terms which come into play are more tangled than what is indicated above.

Note that a normal form approach (in the spirit of [6]: meaning to change $\check{v}$ into $(I + \varepsilon M)\dot{v}$ for some well adjusted operator M), can be tried to get rid of $\mathcal{T}$. However, such a method seems not to succeed. There are always remaining $O(\varepsilon^{-1})$ terms and, all things considered, to deal with the actual diagonal form of $\mathcal{T}$ appears to be more suitable.

• **c** • Finally, consider the full system (23). Our aim is to describe the asymptotic behaviour of the family $\{v_\varepsilon\}_\varepsilon$ on a time scale of the order $t \simeq 1$. To this end, we have to understand the $O(1)$ contributions brought by the singularity $\varepsilon^{-1}\,\mathcal{T} + \varepsilon^{-2}\,\mathcal{Q}$. This singular term is a perturbation of the self adjoint operator $\mathcal{Q}_0 := \varepsilon^{-2}\mu\partial_{\theta\theta} I$. This perturbation is of two types.

- The interactions (at order one) between $\mathcal{Q}_0$ and $\varepsilon^{-1}\mathcal{T}$ turns out to be the source of some creation of diffusion. The mechanism is similar to the one met in the *drift-diffusion* phenomena. Moreover, $\mathcal{Q}_0 + \varepsilon^{-1}\mathcal{T}$ is a diagonal operator. The components of the velocity are decoupled and the discussion deals more with scalar arguments than with vectorial arguments.

- We perturb $\mathcal{Q}_0 + \varepsilon^{-1}\mathcal{T}$ by $\begin{pmatrix} 0 & 0 \\ \partial_\theta h & 0 \end{pmatrix}$ at order 0. A first effect is that $\mathcal{Q}$ is not selfadjoint. It also induced some strong coupling at order 0 between the two components of the velocity. One aspect of the construction is to prove that this strong coupling do not disrupt the production of dissipation. The discussion has to take into account vectorial aspects and one issue is to match the initial data between the slow profile v^s_ε and the fast profile v^f_ε.

Moreover, we have to determine the effects of $\mathcal{LL}$ and $\mathcal{NL}$ which are of two types. First, the presence of $\mathcal{LL}$ and $\mathcal{NL}$ reinforces the coupling. Secondly, it induces nonlinear interactions which are delicate to deal with. In particular, in the critical case $M = 2$, it becomes necessary to exhibit *transparency phenomena* in order to achieve the analysis.

In this article, we propose (Proposition 1.3) and we justify (Theorem 1.4) a complete expansion for the family $\{v_\varepsilon\}_\varepsilon$. It is the occasion to analyze precisely the linear features and the non linear aspects alluded above.

1.4.3 Heuristic arguments for the energy estimates

The variable ${}^t(q^R_\varepsilon, v^R_\varepsilon)$ can be interpreted as the solution of the linearized operator $\mathcal{L}$ at ${}^t(q^a_\varepsilon, v^a_\varepsilon)$ perturbed by some *small* non-linear terms (when ν and M are large enough). It can still be interpreted as the solution of the linearized equation of System (3) at point ${}^t(0,0,h) + (\varepsilon^\nu q^a_\varepsilon, \varepsilon^M v^a_\varepsilon)$

(perturbed b some small non-linear terms). To underline the difficulties to obtain estimates on a strip independent of ε we only consider estimates over $(p_\varepsilon^\ell, u_\varepsilon^\ell)$ solution the linearized system (3) at point ${}^t(0,0,h(\theta))$:

$$\begin{cases} \partial_t p_\varepsilon^\ell & +\varepsilon^{-1} h\, \partial_y p_\varepsilon^\ell & & & = S_0\,, \\ \partial_t u_\varepsilon^{1\ell} & +\varepsilon^{-1} h\, \partial_y u_\varepsilon^{1\ell} & & -\widetilde{\mathcal{P}}_\varepsilon^1 u_\varepsilon^\ell & = S_1\,, \\ \partial_t u_\varepsilon^{2\ell} & +\varepsilon^{-1} h\, \partial_y u_\varepsilon^{2\ell} + & \boxed{\varepsilon^{-2}\partial_\theta h\ u_\varepsilon^{1\ell}} & -\widetilde{\mathcal{P}}_\varepsilon^2 u_\varepsilon^\ell & = S_2\,, \end{cases} \tag{27}$$

for some sources $S := {}^t(S_0, S_1, S_2)$ in $H^\infty(\mathbb{T}\times\mathbb{R}\,;\,\mathbb{R}^3)$. It is a parabolic-hyperbolic system singular in ε.

Purely hyperbolic approach. We first consider that $\mu = \lambda = 0$ so that the dissipation vanishes. We perform classical L^2-estimates on Equation (27). We obtain:

$$\left\| \left(p_\varepsilon^\ell, u_\varepsilon^\ell\right)(t,\cdot)\right\|_{L^2} \lesssim e^{C_\varepsilon t} \sup_{t\in[0,T_\varepsilon^l]} \|S\|_{L^2}\,, \quad \text{with} \quad C_\varepsilon \le C\left(1 + \varepsilon^{-2}\|\partial_\theta h\|_{L^\infty}\right).$$

Yet, it indicates that classical energy estimates only provide a control over the solution for time of order ε^2. In particular for bounded time the solution can exponentially increase with the time t.

Furthermore, let us consider the singular transport equation:

$$\partial_t v + \varepsilon h(\theta)\partial_y v = 0, \qquad v_{|t=0} = v_0\,.$$

The solution is explicit $v(t,\theta,y) = v_0(\theta, y - \varepsilon^{-1} t h(\theta))$. In particular each time we differentiate with respect to θ, we loss a power of ε. Thus, classical Sobolev space are not well suited for the control of this family of solutions. We have to introduce anisotropic Sobolev spaces (defined page 4) for both the velocity and the pressure.

Parabolic-hyperbolic approach. To go further in time, we have to consider the dissipation. The operator $\mathcal{P}_{\varepsilon,}$ is positive and satisfies some coercive estimates. There exists a positive constant c such that for any function $f \in H^1(\mathbb{T}\times\mathbb{R})$,

$$\forall\, \varepsilon \in]0,1], \quad -\left\langle \widetilde{\mathcal{P}}_\varepsilon f, f\right\rangle \ge c\left(\left\|\varepsilon^{-1}\partial_\theta f\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})} + \|\partial_y f\|^2_{L^2(\mathbb{T}\times\mathbb{R})}\right) := \Phi_\varepsilon(\nabla, f). \tag{28}$$

It has two consequences.

- At fixed ε, we should obtain a regularization of the solution. The velocity u_ε^ℓ is in $L_t^2 H^1_{\theta,y}$ (see Inequality (101)). This is the *regularization phenomena*.
- Considering the dependency in ε, the dissipation should also absorb some singular terms. First the squared term can be estimated as follows:

$$\varepsilon^{-1}\left|\varepsilon^{-1}\int_{\mathbb{T}\times\mathbb{R}} \partial_\theta h\, v_\varepsilon^{1l}\, v_\varepsilon^{2l}\, d\theta\, dy\right| \lesssim \varepsilon^{-1}\left(\|h\|^2_{L^\infty}\|v_\varepsilon^l\|^2_{L^2} + \|\varepsilon^{-1}\partial_\theta v_\varepsilon^l\|^2_{L^2}\right).$$

  Thus it only seems to be singular of order one (in ε) instead of being singular of order two (in ε). Furthermore it is also desingularizes the singular transport. We can obtain estimates over the velocity in the classical Sobolev spaces whereas the pressure is still estimated in the anisotropic Sobolev spaces. It indicates that the estimates over the velocity and the pressure have to be done separately.

Here, the addition of the parabolic aspect in the discussion still does not allow us to obtain a control over $(p^\ell_\varepsilon, u^\ell_\varepsilon)$ for time of order one. Some additional arguments are required.

Singular change of unknowns. To keep on desingularizing the term $\varepsilon^{-2}\partial_\theta h\, u^{1\ell}_\varepsilon$ we consider the change of unknowns:

$$\widetilde{q}^\ell_\varepsilon := q^\ell_\varepsilon, \quad \widetilde{u}^{1\ell}_\varepsilon := u^{1\ell}_\varepsilon, \quad \widetilde{u}^{2\ell}_\varepsilon := \varepsilon u^{2\ell}_\varepsilon.$$

The system (27) becomes:

$$\begin{cases} \partial_t \widetilde{p}^\ell_\varepsilon & +\varepsilon^{-1} h\,\partial_y \widetilde{p}^\ell_\varepsilon & & = S_0\,, \\ \partial_t \widetilde{u}^{1\ell}_\varepsilon & +\varepsilon^{-1} h\,\partial_y \widetilde{u}^{1\ell}_\varepsilon & -\mathcal{Q}^1_\varepsilon \widetilde{u}^\ell_\varepsilon & = S_1\,, \\ \partial_t \widetilde{u}^{2\ell}_\varepsilon & +\varepsilon^{-1} h\,\partial_y \widetilde{u}^{2\ell}_\varepsilon + \varepsilon^{-1}\partial_\theta h\, \widetilde{u}^{1\ell}_\varepsilon & -\mathcal{Q}^2_\varepsilon \widetilde{u}^\ell_\varepsilon & = \varepsilon S_2\,. \end{cases} \tag{29}$$

where the operator $\mathcal{Q}_\varepsilon$ is defined in Equation (97). We can notice that:

- The term $\varepsilon^{-2}\partial_\theta h\, u^{1\ell}_\varepsilon$ is desingularized into $\varepsilon^{-1}\partial_\theta h\, \widetilde{u}^{1\ell}_\varepsilon$.
- However, the dissipation is turns into the operator $\mathcal{Q}_\varepsilon$. It can no longer satisfies Inequality (28). Assuming μ is large enough, it is still true (*c.f.* Lemma 3.5).

Thus performing the same estimates for system (29) as the one done in the previous case should lead to a control over $(\widetilde{q}^\ell_\varepsilon, \widetilde{u}^\ell_\varepsilon)$ in L^2-norm for time of order one ($t \approx 1$).

Conclusion. In Section 3, we justify that those heuristic arguments work for the complete System (94). Some technical arguments must be added to deal with the complete System (94). Indeed, it is obviously nonlinear and the pressure and the velocity are coupled. Nonlinear terms has to be studied carefully.

As indicated, the pressure is expected to be controlled in anisotropic Sobolev spaces whereas the velocity is estimated in the classical Sobolev spaces. Thus it indicates that we have to deal with the problem of the velocity and the pressure separately. However those variables are coupled by the term

$$-\frac{C\varepsilon^{2\nu-M-R-2}}{2}\,{}^t(\partial_\theta, \varepsilon\partial_y)\left(q^a_\varepsilon + \varepsilon^R q^R_\varepsilon\right)^2,$$

in Equation (94). The constant ν has to be large enough so that the pressure does not interfere too much with the velocity. Of course an other issue is that the pressure is only estimated in the anisotropic Sobolev spaces. We can go back to the classical Sobolev spaces using the equivalence of norms (92). It has a cost in power of ε for each derivatives to estimate. It explains why we lose $(m+3)$ precision in the definition of w_m (see Equation (18)).

1.4.4 Contents

What follows is divided in two main parts: Section 2 and Section 3.

The Section 2 is devoted to the construction of the approximated solutions $(q^a_\varepsilon, v^a_\varepsilon)$. The first step is to show the Proposition 1.1.

- In this purpose, the paragraph 2.1 deal with the velocity field v^a_ε, that is with the equation $\mathcal{L}^a(\varepsilon, v^a_\varepsilon) = O(\varepsilon^N)$. The limit case $M = 2$ is special because in this situation the non linear terms can interfere at leading order.

- In the paragraph 2.1.5, we are able to exhibit the control (11).

The pressure component q^a_ε is incorporated at the level of subsection 2.2. Then, the complete construction of $(q^a_\varepsilon, v^a_\varepsilon)$ can be achieved in the form of Proposition 1.3.

The Section 3 is concerned with energy estimates. We first state the Proposition of control of the velocity and the pressure. In particular we deduce estimates stated in the Theorem 1.4. In the subsection 3.1, we look at the equations $\mathcal{L}_1$ and $\mathcal{L}_2$. To this end, we crucially need the properties brought by the dissipation. In the Subsection 3.2, we inject the informations which have been obtained at the level of $\mathcal{L}_0$. By this way, we can deduce controls concerning the pressure component.

2 Construction of the approximated solutions

We recall that M is assume to be larger than 2. This section is dedicated to the proof of Propositions 1.1, 1.2 and 1.3.

2.1 Approximated velocity

In this Section, we construct an approximated velocity. Since $M \geq 2$, it follows that non linear effects are present. We are forced to work with the two time scales t and $\varepsilon^{-2}\, t$ together. We construct expansions,

$$v^s_\varepsilon(t,y,\theta) \,=\sum_{k=0}^{N+1} \varepsilon^k\, v^s_k(t,y,\theta)\,, \qquad v^f_\varepsilon\big(\frac{t}{\varepsilon^2},y,\theta\big) \,=\sum_{k=0}^{N+1} v^f_k\big(\frac{t}{\varepsilon^2},y,\theta\big)\,, \tag{30}$$

and plug the expression $v^s_\varepsilon + v^f_\varepsilon$ into $\mathcal{L}^a$ at the level of (4). We obtain:

$$\begin{aligned}
&\partial_t v^s_\varepsilon(t,.) + \varepsilon^{-1}\, h\, \partial_y v^s_\varepsilon(t,.) + \varepsilon^{M-2}\big(v^{1s}_\varepsilon(t,.)\partial_\theta v^s_\varepsilon(t,.) + \varepsilon v^{2s}_\varepsilon(t,.)\partial_y v^s_\varepsilon(t,.)\big)\\
&\qquad + \varepsilon^{-2}\ {}^t\big(0,\partial_\theta h\, v^{1s}_\varepsilon(t,.)\big) - \widetilde{\mathcal{P}}_\varepsilon v^s_\varepsilon(t,.)\\
&+\varepsilon^{-2}\,\partial_\tau v^f_\varepsilon(t/\varepsilon^2,.) + \varepsilon^{-1} h\,\partial_y v^f_\varepsilon(t/\varepsilon^2,.) + \varepsilon^{M-2}\, v^{1f}_\varepsilon(t/\varepsilon^2,.)\,\partial_\theta v^f_\varepsilon(t/\varepsilon^2,.)\\
&\quad + \varepsilon^{M-1}\, v^{2f}_\varepsilon(t/\varepsilon^2,.)\,\partial_y v^f_\varepsilon(t/\varepsilon^2,.) + \varepsilon^{-2}\ {}^t\big(0,\partial_\theta h\, v^{1f}_\varepsilon(t/\varepsilon^2,.)\big) - \widetilde{\mathcal{P}}_\varepsilon v^f_\varepsilon(t/\varepsilon^2,.)\\
&\quad + \varepsilon^{M-2}\big(v^{1s}_\varepsilon(t,.)\partial_\theta v^f_\varepsilon(t/\varepsilon^2,.) + \varepsilon v^{2s}_\varepsilon(t,.)\partial_y v^f_\varepsilon(t/\varepsilon^2,.)\big)\\
&\quad + \varepsilon^{M-2}\, v^{1f}_\varepsilon(t/\varepsilon^2,.)\,\partial_\theta v^s_\varepsilon(t,.) + \varepsilon^{M-1}\, v^{2f}_\varepsilon(t/\varepsilon^2,.)\,\partial_y v^s_\varepsilon(t,.)\,.
\end{aligned}\tag{31}$$

Fix any time $T\in\mathbb{R}^*_+$. Define

$$\mathcal{L}^{as}(\varepsilon,v^s_\varepsilon) := \partial_t v^s_\varepsilon + \varepsilon^{-1}\, h\,\partial_y v^s_\varepsilon + \varepsilon^{M-2}\big(v^{1s}_\varepsilon\partial_\theta v^s_\varepsilon + \varepsilon v^{2s}_\varepsilon\partial_y v^s_\varepsilon\big) + \varepsilon^{-2}\ {}^t\big(0,\partial_\theta h\, v^{1s}_\varepsilon\big) - \widetilde{\mathcal{P}}_\varepsilon v^s_\varepsilon\,. \tag{32}$$

Recall that $v^f_\varepsilon \in \mathcal{E}^\infty_\delta$ is assumed to be exponentially decreasing with respect to $\tau\in\mathbb{R}_+$. Since

$$e^{-\delta\, t/\varepsilon^2} \,=\, O\big((\varepsilon^2/t)^\infty\big) \,=\, O(\varepsilon^N)\,, \qquad \forall\,(t,N)\in\,]0,T]\times\mathbb{N}\,,$$

when looking at the Equation (31) for times $t\in\,]0,T]$ with in view a precision of the size $O(\varepsilon^N)$, all terms involving v^f_ε can be neglected. Now, the idea is simply to extend this (relaxed) smallness requirement on the whole interval $[0,T]$. Briefly, we seek v^s_ε so that

$$\mathcal{L}^{as}(\varepsilon,v^s_\varepsilon) \,=\, O(\varepsilon^N)\,, \qquad t\in[0,T]\,. \tag{33}$$

The Equation (33) can be completed with some initial data

$$v_\varepsilon^s(0,\theta,y) \, = \, v_\varepsilon^{s0}(\theta,y) \, = \, \sum_{k=0}^{N+1} \varepsilon^k \; v_k^{s0}(\theta,y) \, . \tag{34}$$

Clearly, it suffices to specify v_ε^{s0} to determine what is $v_\varepsilon^s(t,\cdot)$ for $t \in [0,T]$, by solving the Cauchy problem (33)-(34). Now, in order to select v_ε^{s0} conveniently, we have to take into account what happens for small times, in a boundary layer of size ε^2 near $t=0$. To understand why, just come back to the study of (31) for $t \simeq \varepsilon^2$ or $\tau \simeq 1$. Then, the contributions brought by v_ε^f can no more be neglected. Considering (31) with the information (33) in mind, it seems natural to impose

$$\mathcal{L}^{af}(\varepsilon, v_\varepsilon^f, v_\varepsilon^s) \, = \, O(\varepsilon^N) \, , \qquad \tau \in [0,1] \tag{35}$$

where we have introduced

$$\begin{aligned} \mathcal{L}^{af}(\varepsilon, v_\varepsilon^f, v_\varepsilon^s) := \varepsilon^{-2} \partial_\tau v_\varepsilon^f(\tau,.) + \varepsilon^{-1} h \, \partial_y v_\varepsilon^f(\tau,.) - \widetilde{\mathcal{P}}_\varepsilon v_\varepsilon^f(\tau,.) \\ + \varepsilon^{M-2} \left(v_\varepsilon^{1f}(\tau,.) \partial_\theta v_\varepsilon^f(\tau,.) + \varepsilon v_\varepsilon^{2f}(\tau,.) \partial_y v_\varepsilon^f(\tau,.) \right) + {}^t \left(0, \partial_\theta h \, v_\varepsilon^{1f}(\tau,.) \right) \\ + \varepsilon^{M-2} \left(v_\varepsilon^{1s}(\varepsilon^2\tau,.) \partial_\theta v_\varepsilon^f(\tau,.) + \varepsilon v_\varepsilon^{2s}(\varepsilon^2\tau,.) \partial_y v_\varepsilon^f(\tau,.) \right) \\ + \varepsilon^{M-2} \left(v_\varepsilon^{1f}(\tau,.) \partial_\theta v_\varepsilon^s(\varepsilon^2\tau,.) + \varepsilon v_\varepsilon^{2f}(\tau,.) \partial_y v_\varepsilon^s(\varepsilon^2\tau,.) \right) \, . \end{aligned} \tag{36}$$

Assume that the data v_ε^{s0} is known. The Cauchy problem (33)-(34) furnishes $v_\varepsilon^s(t,\cdot)$ for $t \in [0,T]$. In particular, it gives access to all derivatives $(\partial_\tau)^l v_\varepsilon^{*s}(0,.)$ with $l \in \mathbb{N}$. Therefore, we can go further in the analysis by replacing in $\mathcal{L}^{af}(\varepsilon, v_\varepsilon^f, v_\varepsilon^s)$ all the expressions $v_\varepsilon^{*s}(\varepsilon^2\tau,.)$ by their corresponding Taylor expansions (say up to the order $N-1$) near $t=0$. As long as $\tau \in \mathbb{R}$ is fixed, this operation is justified. From now on, we look at

$$\mathcal{L}^{aft}(\varepsilon, v_\varepsilon^f) \, = \, O(\varepsilon^N) \, , \qquad \tau \in \mathbb{R}_+ \tag{37}$$

where the definition of $\mathcal{L}^{aft}$ is

$$\begin{aligned} \mathcal{L}^{aft}(\varepsilon, v_\varepsilon^f) : = \; & \varepsilon^{-2} \, \partial_\tau v_\varepsilon(\tau,.) + \varepsilon^{-1} h \, \partial_y v_\varepsilon(\tau,.) - \widetilde{\mathcal{P}}_\varepsilon v_\varepsilon(\tau,.) \\ & + \varepsilon^{M-2} \left(v_\varepsilon^1(\tau,.) \partial_\theta v_\varepsilon(\tau,.) + \varepsilon v_\varepsilon^2(\tau,.) \partial_y v_\varepsilon(\tau,.) \right) + \varepsilon^{-2} \; {}^t \left(0, \partial_\theta h \, v_\varepsilon^1(\tau,.) \right) \\ & + \varepsilon^{M-2} \left(\sum_{l=0}^{N-1} \frac{(\varepsilon^2\tau)^l}{l!} (\partial_t)^l v_\varepsilon^{1s}(0,.) \partial_\theta v_\varepsilon(\tau,.) + \varepsilon \sum_{l=0}^{N-1} \frac{(\varepsilon^2\tau)^l}{l!} (\partial_t)^l v_\varepsilon^{2s}(0,.) \partial_y v_\varepsilon(\tau,.) \right) \\ & + \varepsilon^{M-2} \left(v_\varepsilon^1(\tau,.) \sum_{l=0}^{N-1} \frac{(\varepsilon^2\tau)^l}{l!} (\partial_t)^l \partial_\theta v_\varepsilon^s(0,.) + \varepsilon \, v_\varepsilon^2(\tau,.) \sum_{l=0}^{N-1} \frac{(\varepsilon^2\tau)^l}{l!} (\partial_t)^l \partial_y v_\varepsilon^s(0,.) \right) . \end{aligned} \tag{38}$$

To be coherent with (4), we have to impose

$$v_\varepsilon^f(0,\cdot) \, = \, v_\varepsilon^{f0}(\cdot) \, := \, (v_\varepsilon^a - v_\varepsilon^s)(0,\cdot) \, , \tag{39}$$

where $v_\varepsilon^a(0,\cdot)$ and $v_\varepsilon^s(0,\cdot)$ are prescribed as indicated in lines (10) and (34). Now, we can recover some $v_\varepsilon^f(\tau,\cdot)$ for $\tau \in \mathbb{R}_+$ by solving the Cauchy problem (37)-(39).

The difficulty comes from the condition $v_\varepsilon^f \in \mathcal{E}_\delta^\infty$. Nothing guarantees that the criterion $v_\varepsilon^f \in \mathcal{E}_\delta^\infty$ can be verified for some well-chosen v_ε^{s0}. To show the existence and the uniqueness of such a data v_ε^{s0} is in fact what matters. The extraction of an adequate function v_ε^{s0} is clarified in the construction described below.

For the sake of brevity, for $k \geq M$, introduce the following notations:

$$\mathcal{J}(M,k) := \left\{ (i,j) \in [\![0, N+1]\!]^2 \; ; \; i + j = k - M \right\} \,, \tag{40}$$

$$\mathcal{I}(M,k) := \left\{ (i,j,l) \in [\![0, N+1]\!]^2 \times \; [\![0, N-1]\!] \; ; \; i + j + 2l = k - M \right\} \,. \tag{41}$$

For $k < M$, we set $\mathcal{J}(M,k) = \emptyset$ and $\mathcal{I}(M,k) = \emptyset$. We also adopt the conventions $v_k^s \equiv v_k^f \equiv 0$ for $k = -3$, $k = -2$ and $k = -1$. Let us now go into the details of the BKW calculus.

The first step is to inject some expansion v_ε^s like in (30) into the Equation (33). By this way, we can obtain a cascade of equations concerning the unknowns $v_k^s = {}^t(v_{1,k}^s, v_{2,k}^s)$. More precisely, for $k \in [\![0, N+3]\!]$, we have to consider

$$\begin{aligned} \partial_t v_{1,k-2}^s + h\, \partial_y v_{1,k-1}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{1,i}^s \partial_\theta v_{1,j}^s + \sum_{(i,j) \in \mathcal{J}(M+1,k)} v_{2,i}^s \partial_y v_{1,j}^s & \\ = \mu \partial_{\theta\theta} v_{1,k}^s + \mu \partial_{yy} v_{1,k-2}^s + \lambda \partial_{\theta\theta} v_{1,k-1}^s + \lambda \partial_{\theta y} v_{2,k-2}^s \,, & \end{aligned} \tag{42a}$$

$$\begin{aligned} \partial_t v_{2,k-2}^s + h\, \partial_y v_{2,k-1}^s + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{1,i}^s \partial_\theta v_{2,j}^s + \sum_{(i,j) \in \mathcal{J}(M+1,k)} v_{2,i}^s \partial_y v_{2,j}^s + \partial_\theta h\, v_{1,k}^s & \\ = \mu \partial_{\theta\theta} v_{2,k}^s + \mu \partial_{yy} v_{2,k-2}^s + \lambda \partial_{\theta y} v_{1,k-2}^s + \lambda \partial_{yy} v_{2,k-3}^s \,. & \end{aligned} \tag{42b}$$

The next step is to plug some expansion v_ε^f like in (30) into the Equation (37). By this way, we can obtain a cascade of equations concerning the unknowns $v_k^f = {}^t(v_{1,k}^f, v_{2,k}^f)$. More precisely, for $k \in [\![0, N+1]\!]$, we have to consider

$$\begin{aligned} \partial_\tau v_{1,k}^f + h\, \partial_y v_{1,k-1}^f + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{1,i}^f \partial_\theta v_{1,j}^f + \sum_{(i,j) \in \mathcal{J}(M+1,k)} v_{2,i}^f \partial_y v_{1,j}^f & \\ + \sum_{(i,j,l) \in \mathcal{I}(M,k)} \left(v_{1,i}^f \, (\partial_t)^l \, (\partial_\theta v_{1,j}^s)(0) + (\partial_t)^l \, (v_{1,i}^s)(0) \partial_\theta v_{1,j}^f \right) \frac{\tau^l}{l!} & \\ + \sum_{(i,j,l) \in \mathcal{I}(M+1,k)} \left(v_{2,i}^f \, (\partial_t)^l \, (\partial_y v_{1,j}^s)(0) + (\partial_t)^l \, (v_{2,i}^s)(0) \partial_y v_{1,j}^f \right) \frac{\tau^l}{l!} & \\ = \mu \partial_{\theta\theta} v_{1,k}^f + \mu \partial_{yy} v_{1,k-2}^f + \lambda \partial_{\theta\theta} v_{1,k-1}^f + \lambda \partial_{\theta y} v_{2,k-2}^f \,, & \end{aligned} \tag{43a}$$

$$\begin{aligned} \partial_\tau v_{2,k}^f + h\, v_{2,k-1}^f + \partial_\theta h\, v_{1,k}^f + \sum_{(i,j) \in \mathcal{J}(M,k)} v_{1,i}^f \partial_\theta v_{2,j}^f + \sum_{(i,j) \in \mathcal{J}(M,k+1)} v_{2,i}^f \partial_\theta v_{2,j}^f & \\ + \sum_{(i,j,l) \in \mathcal{I}(M,k)} \left(v_{1,i}^f \, (\partial_t)^l \, (\partial_\theta v_{2,j}^s)(0) + (\partial_t)^l \, (v_{1,i}^s)(0) \partial_\theta v_{2,j}^f \right) \frac{\tau^l}{l!} & \\ + \sum_{(i,j,l) \in \mathcal{I}(M+1,k)} \left(v_{2,i}^f \, (\partial_t)^l \, (\partial_y v_{2,j}^s)(0) + (\partial_t)^l \, (v_{2,i}^s)(0) \partial_y v_{2,j}^f \right) \frac{\tau^l}{l!} & \\ = \mu \partial_{\theta\theta} v_{2,k}^f + \mu \partial_{yy} v_{2,k-2}^f + \lambda \partial_{\theta y} v_{1,k-2}^f + \lambda \partial_{yy} v_{2,k-3}^f \,. & \end{aligned} \tag{43b}$$

In view of (10), we can associate (42) and (43) with initial data v_k^{s0} and v_k^{f0} satisfying the restriction:

$$v_k^s(0,\theta,y) + v_k^f(0,\theta,y) \; = \; v_k^0(\theta,y) \,, \qquad \forall \, k \in [\![0, N+1]\!] \,. \tag{44}$$

Proposition 2.1. *[Solving (42) and (43) together with (44) and the condition $v_k^f \in \mathcal{E}_\delta^\infty$] Fix a time $T \in \mathbb{R}_+^*$, a number $\delta \in]0, \mu[$ and, for all $k \in [\![0, \dots, N+1]\!]$, functions $v_k^0 \in H^\infty(\mathbb{T} \times \mathbb{R})$. Then, the conditions (42$\star$), (43$\star$) and (44) have a unique solution such that*

$$(v_k^s, v_k^f) \in \mathcal{H}_T^\infty \times \mathcal{E}_\delta^\infty, \qquad \forall\, k \in [\![0, N+1]\!] . \tag{45}$$

Moreover, the component v_k^s can be identified through the homogenized Equation ().

The proof relies on some induction on the size of N, based on the following hypothesis of induction:

$$\mathcal{HN}(N) \; : \; \text{" The Proposition 2.1 is verified up to the integer } N \text{".} \tag{46}$$

To go from N up to $N+1$, we will need a succession of lemmas which are produced below. Before going into the details of the analysis, we give below a brief description of what happens depending on the choice of M.

- When $M \geq 3$, the non linear terms are rather small, the construction is somehow linear.
- When $M = 2$, the non linearity becomes critical and a few arguments must be added. For instance, if we write the Equation (42a) for $k = 2$, we can notice a Burgers' term

$$\partial_t v_{1,0}^s + h\, \partial_y v_{1,1}^s + v_{1,0}^s\, \partial_\theta v_{1,0}^s = \mu\, \partial_{\theta\theta} v_{1,2}^s + \mu\, \partial_{yy} v_{1,0}^s + \lambda \partial_{\theta\theta} v_{1,1}^s + \lambda \partial_{\theta y} v_{2,0}^s \tag{47}$$

  and also two contributions $\partial_{\theta\theta} v_{1,2}^s$ and $\partial_{\theta\theta} v_{1,1}^s$ to be calculated in function of $v_{1,0}^s$, with apparently a non linear dependence with respect to $v_{1,0}^s$.
- When $M = 2$ again, another effect of the non linear interactions is the apparition at the level of (50) below together with (62), of a source term $S_k^{nl /\!/}$ which can depend on v_k^s.
- However, there are *transparency phenomena* at work which come from the initialization procedure. Indeed, knowing that $v_k^s \equiv 0$ for $k \in \{-3, -2, -1\}$, the Equation (42) in the case $k = 0$ and $M \geq 2$ reduces to $\mu\, \partial_{\theta\theta} v_0^s = 0$. In other words, we have to impose

$$v_{1,0}^{s\perp} \equiv (I - \Pi)\, v_{1,0}^s \equiv 0 . \tag{48}$$

  It follows that $v_{1,0}^s\, \partial_\theta v_{1,0}^s \equiv 0$. In the same way, all other apparent non linear contributions will disappear. Therefore, the remaining term $\Pi v_{1,0}^s$ can be determined apart without seeing any non linear effect.

2.1.1 Technical Lemmas

A consequence of the (point) spectral gap. The system (43) is made of two evolution equations of parabolic type, based on $\partial_\tau - \mu \partial_{\theta\theta}$. This falls under the following framework.

Lemma 2.2. *[Fast decreasing under a polarization condition] Let $m \in \mathbb{N}$ and $\delta \in]0, \mu[$. Select $w_0 \in H^{m+2}(\mathbb{T} \times \mathbb{R})$ and $S_0 \in (\mathcal{E}_\delta^{m+2})^\perp(\mathbb{T} \times \mathbb{R})$, that is such that $\Pi S_0 = 0$. Consider the initial value problem:*

$$\partial_\tau w - \mu\, \partial_{\theta\theta} w = S_0 , \qquad w_{|\tau=0} = w_0 . \tag{49}$$

For all $T \in \mathbb{R}_+$, there is a unique solution $w \in C^1\big([0,T]; H^m(\mathbb{T} \times \mathbb{R})\big)$ to the Cauchy problem (49). Moreover, if the initial data is well prepared in the sense that $\Pi w_0 = 0$, then $\Pi w = 0$ for all $t \in [0,T]$ and $w \in \mathcal{E}_\delta^m(\mathbb{T} \times \mathbb{R})$.

The proof of Lemma 2.2 is very easy. It will not be detailed.

Interpretation of the system (42). The Lemma below is intended to look at the system (42) otherwise. Indeed, there is a difficulty when dealing with (42) since the knowledge of v^s_{k-2} seems to require the determination of v^s_{k-1} and v^s_k, that is the identification of terms v^s_j with indices j *greater* than $k-2$. An important remark is that such a dependence can disappear when the projector Π is applied. The possible dependence in the terms v^s_j are recorded in the source terms $S^{nl/\!/}_k$ (with explicit formulas). For this occasion, the two cases $M \geq 3$ and $M = 2$ must be distinguished. This fact can be formulated in the following way.

Lemma 2.3. *[Non-linear homogenization] Assume that the functions v^s_k with $k \in [\![0, N+3]\!]$ are solutions of the system (42). Then, for all $k \in [\![0, N+1]\!]$, the part $\Pi v^s_k = {}^t(\Pi v^s_{1,k}, \Pi v^s_{2,k})$ is a solution of*

$$
\left\{
\begin{aligned}
&\partial_t \Pi v^s_{1,k} - \Big(\mu + \tfrac{1}{\mu}\,\Pi((\partial_\theta^{-1} h)^2)\Big)\,\partial_{yy}\Pi v^s_{1,k} = S^{l/\!/}_{1,k} + S^{nl/\!/}_{1,k} + P^{l\perp}_1 (I-\Pi) v^s_{1,k}\,,\\
&\partial_t \Pi v^s_{2,k} - \Big(\mu + \tfrac{1}{\mu}\,\Pi((\partial_\theta^{-1} h)^2)\Big)\,\partial_{yy}\Pi v^s_{2,k} = S^{l/\!/}_{2,k} + S^{nl/\!/}_{2,k} + P^{l\perp}_2 (I-\Pi) v^s_{1,k}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + P^{l/\!/}_2 \Pi v^s_{1,k} + Q^{l\perp}_2 (I-\Pi) v^s_{2,k}\,.
\end{aligned}
\right.
\tag{50}
$$

In the above system (50), the four operators $P^{l\perp}_1$, $P^{l\perp}_2$, $P^{l/\!/}_2$ and $Q^{l\perp}_2$, as well as the source term $S^{l/\!/}_k = {}^t(S^{l/\!/}_{1,k}, S^{l/\!/}_{2,k})$, are defined along lines (51)-$\cdots$-(58). On the other hand, the contribution $S^{nl/\!/}_k = {}^t(S^{nl/\!/}_{1,k}, S^{nl/\!/}_{2,k})$ is given by (59)-(60).

Proof of Lemma 2.3. The matter is to identify the contributions brought by the non linear terms. Let us project () according to $\mathcal{V}^{/\!/}$. Since $\mathcal{J}(M+1,k+2) \equiv \mathcal{J}(M,k+1)$, this yields

$$
\begin{aligned}
&\partial_t \Pi v^s_{1,k} + \Pi\left(h\,\partial_y v^s_{1,k+1}\right) + \Pi\left(\sum_{(i,j)\in\mathcal{J}(M,k+2)} v^s_{1,i}\,\partial_\theta v^s_{1,j}\right)\\
&\qquad + \Pi\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{2,i}\,\partial_y v^s_{1,j}\right) = \mu\,\partial_{yy}\Pi v^s_{1,k}\,,\\
&\partial_t \Pi v^s_{2,k} + \Pi\left(h\,\partial_y v^s_{2,k+1}\right) + \Pi\left(\sum_{(i,j)\in\mathcal{J}(M,k+2)} v^s_{1,i}\,\partial_\theta v^s_{2,j}\right)\\
&\qquad + \Pi\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{2,i}\,\partial_y v^s_{2,j}\right) + \Pi\left(\partial_\theta h\, v^s_{1,k+2}\right) = \mu\,\partial_{yy}\Pi v^s_{2,k} + \lambda\,\partial_{yy}\Pi v^s_{2,k-1}\,.
\end{aligned}
$$

In what follows, we will use the system (42) and many integrations by parts in order to interpret $\Pi(h\,\partial_y v^s_{1,k+1})$, $\Pi\left(h\,\partial_y v^s_{2,k+1}\right)$ and $\Pi\left(\partial_\theta h\, v^s_{1,k+2}\right)$. The goal is to show that these quantities can be expressed in terms of the v^s_j with $j \in [\![0, k]\!]$.

$\circ$ Study of $\Pi(h\,\partial_y v^s_{1,k+1})$. Exploiting (42a) with $k+1$ in place of k, we can deduce

$$\begin{aligned}
\Pi(h\,\partial_y v^s_{1,k+1}) &= \Pi\big(\partial_{\theta\theta}^{-2}(h)\,\partial_y\partial_{\theta\theta} v^s_{1,k+1}\big)\,,\\
&= \frac{1}{\mu}\,\Pi\big(\partial_{\theta\theta}^{-2}(h)\;\partial_y(\partial_t v^s_{1,k-1} - \mu\partial_{yy} v^s_{1,k-1} - \lambda\partial_{\theta y} v^s_{2,k-1})\big)\\
&+ \frac{1}{\mu}\,\Pi\left(\partial_{\theta\theta}^{-2}(h)\;\partial_y\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{1,i}\,\partial_\theta v^s_{1,j} + \sum_{(i,j)\in\mathcal{J}(M,k)} v^s_{2,i}\,\partial_y v^s_{1,j}\right)\right)\\
&+ \frac{1}{\mu}\,\Pi\big(h\,\partial_{\theta\theta}^{-2}(h)\,\partial_{yy} v^s_{1,k}\big) - \frac{\lambda}{\mu}\,\Pi\big(\partial_{\theta\theta}^{-2}(h)\,\partial_{y\theta\theta} v^s_{1,k}\big)\,.
\end{aligned}$$

Recalling (6) and (7), we have to deal with

$$\begin{aligned}
\Pi(h\,\partial_y v^s_{1,k+1}) = &-\frac{1}{\mu}\,\Pi\left((\partial_\theta^{-1}h)^2\right)\partial_{yy}\Pi v^s_{1,k} - S^{l/\!/}_{1,k} - P_1^{l\perp}(I-\Pi)v^s_{1,k}\\
&+ \frac{1}{\mu}\,\Pi\left(\partial_{\theta\theta}^{-2}(h)\;\partial_y\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{1,i}\,\partial_\theta v^s_{1,j} + \sum_{(i,j)\in\mathcal{J}(M,k)} v^s_{2,i}\,\partial_y v^s_{1,j}\right)\right)\,,
\end{aligned}$$

with the conventions,

$$S^l_{1,k} := -\frac{1}{\mu}\,\partial_{\theta\theta}^{-2}(h)\,\partial_y\left(\partial_t v^s_{1,k-1} - \mu\partial_{yy} v^s_{1,k-1} - \lambda\partial_{\theta y} v^s_{2,k-1}\right)\,, \tag{51}$$

$$P^l_1 f := \frac{\lambda}{\mu}\,\Pi\left(\partial_{\theta\theta}^{-2}(h)\,\partial_{\theta\theta y} f\right) - \frac{1}{\mu}\,\Pi\left(h\,\partial_{\theta\theta}^{-2}(h)\,\partial_{yy} f\right)\,. \tag{52}$$

$\circ$ Study of $\Pi(h\,\partial_y v^s_{2,k+1})$. Exploiting (42b) with $k+1$ in place of k, we can derive

$$\begin{aligned}
\Pi\left(h\,\partial_y v^s_{2,k+1}\right) &= \Pi\big(\partial_{\theta\theta}^{-2}(h)\,\partial_y\partial_{\theta\theta} v^s_{2,k+1}\big)\,,\\
&= \frac{1}{\mu}\,\Pi\left(\partial_{\theta\theta}^{-2}(h)\,\partial_y\left(\partial_t v^s_{2,k-1} - \mu\partial_{yy} v^s_{2,k-1} - \lambda\partial_{\theta y} v^s_{1,k-1} - \lambda\partial_{yy} v^s_{2,k-2}\right)\right)\\
&+ \frac{1}{\mu}\,\Pi\left(\partial_{\theta\theta}^{-2}(h)\;\partial_y\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{1,i}\,\partial_\theta v^s_{2,j} + \sum_{(i,j)\in\mathcal{J}(M,k)} v^s_{2,i}\,\partial_y v^s_{2,j}\right)\right)\\
&+ \frac{1}{\mu}\,\Pi\big(\partial_{\theta\theta}^{-2}(h)\;\partial_y(h\,\partial_y v^s_{2,k} + \partial_\theta h\,v^s_{1,k+1})\big)\,.
\end{aligned}$$

We come back to the Equation (42a) in order to change the last term in this sum. This yields

$$\begin{aligned}
&\frac{1}{\mu}\,\Pi\big(\partial_{\theta\theta}^{-2}(h)\,\partial_y(\partial_\theta h\,v^s_{1,k+1})\big) = \frac{1}{\mu}\Pi\left(\partial_\theta h\,\partial_{\theta\theta}^{-2}(h)\,\partial_y(I-\Pi)v^s_{1,k+1}\right)\,,\\
&= \frac{1}{\mu^2}\,\Pi\left(\partial_\theta h\,\partial_{\theta\theta}^{-2}(h)\,\partial_y\partial_{\theta\theta}^{-2}(I-\Pi)\left(\partial_t v^s_{1,k-1} - \mu\partial_{yy} v^s_{1,k-1} - \lambda\partial_{\theta y} v^s_{2,k-1}\right)\right)\\
&+ \frac{1}{\mu^2}\,\Pi\left(\partial_\theta h\,\partial_{\theta\theta}^{-2}(h)\,\partial_y\partial_{\theta\theta}^{-2}(I-\Pi)\left(h\,\partial_y v^s_{1,k} - \lambda\partial_{\theta\theta} v^s_{1,k}\right)\right)\\
&+ \frac{1}{\mu^2}\,\Pi\left(\partial_\theta h\,\partial_{\theta\theta}^{-2}(h)\,\partial_y\partial_{\theta\theta}^{-2}(I-\Pi)\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{1,i}\,\partial_\theta v^s_{1,j} + \sum_{(i,j)\in\mathcal{J}(M,k)} v^s_{2,i}\,\partial_y v^s_{1,j}\right)\right)\,.
\end{aligned}$$

This amounts to the same thing as

$$\Pi(h\,\partial_y v^s_{2,k+1}) = -\frac{1}{\mu}\,\Pi\left((\partial_\theta^{-2}h)^2\right)\partial_{yy}\Pi v^s_{2,k} - S^{l1/\!/}_{2,k} - P^{l1\perp}_2\,(I-\Pi)v^s_{1,k} - P^{l1/\!/}_2\,\Pi v^s_{1,k}$$
$$- Q^{l1\perp}_2\,(I-\Pi)v^s_{2,k} + \frac{1}{\mu}\,\Pi\left(\partial_{\theta\theta}^{-2}(h)\,\partial_y\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{1,i}\,\partial_\theta v^s_{1,j} + \sum_{(i,j)\in\mathcal{J}(M,k)} v^s_{2,i}\,\partial_y v^s_{1,j}\right)\right)$$
$$+ \frac{1}{\mu^2}\,\Pi\left(\partial_\theta h\,\partial_{\theta\theta}^{-2}(h)\,\partial_y\partial_{\theta\theta}^{-2}(I-\Pi)\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{1,i}\,\partial_\theta v^s_{1,j} + \sum_{(i,j)\in\mathcal{J}(M,k)} v^s_{2,i}\,\partial_y v^s_{1,j}\right)\right).$$

with the notations

$$S^{l1}_{2,k} := -\frac{1}{\mu}\,\partial_{\theta\theta}^{-2}(h)\,\partial_y\left(\partial_t v^s_{2,k-1} - \mu\partial_{yy}v^s_{2,k-1} - \lambda\partial_{\theta y}v^s_{1,k-1} - \lambda\partial_{yy}v^s_{2,k-2}\right)$$
$$-\frac{1}{\mu^2}\,\partial_\theta h\,\partial_{\theta\theta}^{-2}(h)\,\partial_y\partial_{\theta\theta}^{-2}(I-\Pi)\left(\partial_t v^s_{1,k-1} - \mu\partial_{yy}v^s_{1,k-1} - \lambda\partial_{\theta y}v^s_{2,k-1}\right), \tag{53}$$

$$P^{l1}_2 f := -\Pi\left(\frac{1}{\mu^2}\,\partial_\theta h\,\partial_{\theta\theta}^{-2}(h)\,\partial_y\partial_{\theta\theta}^{-2}(I-\Pi)\,(h\partial_y f - \lambda\partial_{\theta\theta}f)\right), \tag{54}$$

$$Q^{l1}_2 f := -\frac{1}{\mu}\,\Pi\left(h\,\partial_{\theta\theta}^{-2}(h)\,\partial_{yy}f\right). \tag{55}$$

∘ There remains to compute $\Pi(\partial_\theta h\,v^s_{1,k+2})$. This is again (42a) with this time $k+2$ in place of k.

$$\Pi\left(\partial_\theta h\,v^s_{1,k+2}\right) = \Pi\left(\partial_\theta^{-1}(h)\,\partial_{\theta\theta}v^s_{1,k+2}\right),$$
$$= \frac{1}{\mu}\,\Pi\left(\partial_\theta^{-1}(h)\left(\partial_t v^s_{1,k} - \mu\partial_{yy}v^s_{1,k} - \lambda\partial_{\theta y}(I-\Pi)v^s_{2,k}\right)\right)$$
$$+ \frac{1}{\mu}\,\Pi\big(\partial_\theta^{-1}(h)\,(h\,\partial_y v^s_{1,k+1} - \lambda\partial_{\theta\theta}v^s_{1,k+1})\big)$$
$$+ \frac{1}{\mu}\,\Pi\left(\partial_\theta^{-1}(h)\left(\sum_{(i,j)\in\mathcal{J}(M,k+2)} v^s_{1,i}\,\partial_\theta v^s_{1,j} + \sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{2,i}\,\partial_y v^s_{1,j}\right)\right).$$

We want to remove the presence of $v^s_{1,k+1}$. The relation $\Pi(h\,\partial_\theta^{-1}h) = 0$ allows to write

$$\Pi\big(\partial_\theta^{-1}(h)\,(h\,\partial_y v^s_{1,k+1} - \lambda\partial_{\theta\theta}v^s_{1,k+1})\big) \;=\; \Pi\big(\partial_\theta^{-1}(h)\,(h\,\partial_y - \lambda\partial_{\theta\theta})\,\partial_{\theta\theta}^{-2}\,(I-\Pi)\,\partial_{\theta\theta}v^s_{1,k+1})\big).$$

The part $\partial_{\theta\theta} v^s_{1,k+1}$ can be extracted from the Equation (42a) with $k+1$ in place of k. We find

$$\begin{aligned}
&\Pi\left(\partial_\theta h\, v^s_{1,k+2}\right) = \frac{1}{\mu}\,\Pi\left(\partial_\theta^{-1}(h)\ \left(\partial_t v^s_{1,k} - \mu\partial_{yy} v^s_{1,k} - \lambda\partial_{\theta y}(I-\Pi)v^s_{2,k}\right)\right)\\
&\quad + \frac{1}{\mu^2}\,\Pi\left(\partial_\theta^{-1}(h)\,(h\,\partial_y - \lambda\partial_{\theta\theta})\partial_{\theta\theta}^{-2}(I-\Pi)\left(\partial_t v^s_{1,k-1} - \mu\partial_{yy} v^s_{1,k-1} - \lambda\partial_{\theta y} v^s_{2,k-1}\right)\right)\\
&\quad + \frac{1}{\mu^2}\,\Pi\left(\partial_\theta^{-1}(h)\,(h\,\partial_y - \lambda\partial_{\theta\theta})\partial_{\theta\theta}^{-2}(I-\Pi)\left(h\,\partial_y v^s_{1,k} - \lambda\partial_{\theta\theta} v^s_{1,k}\right)\right)\\
&\quad + \frac{1}{\mu^2}\,\Pi\left(\partial_\theta^{-1}(h)(h\,\partial_y - \lambda\partial_{\theta\theta})\partial_{\theta\theta}^{-1}(I-\Pi)\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{1,i}\,\partial_\theta v^s_{1,j} + \sum_{(i,j)\in\mathcal{J}(M,k)} v^s_{2,i}\,\partial_y v^s_{1,j}\right)\right)\\
&\quad + \frac{1}{\mu}\,\Pi\left(\partial_\theta^{-1}(h)\left(\sum_{(i,j)\in\mathcal{J}(M,k+2)} v^s_{1,i}\,\partial_\theta v^s_{1,j} + \sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{2,i}\,\partial_y v^s_{1,j}\right)\right).
\end{aligned}$$

This yields to

$$\begin{aligned}
&\Pi\left(\partial_\theta h\, v^s_{1,k+2}\right) = -S^{l2/\!/}_{2,k} - P^{l2\perp}_2\,(I-\Pi)v^s_{1,k} - P^{l2/\!/}_2\,\Pi v^s_{1,k} - Q^{l2\perp}_2\,(I-\Pi)v^s_{2,k}\\
&\quad + \frac{1}{\mu^2}\,\Pi\left(\partial_\theta^{-1}(h)\,(h\,\partial_y - \lambda\partial_{\theta\theta})\partial_{\theta\theta}^{-2}(I-\Pi)\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{1,i}\,\partial_\theta v^s_{1,j} + \sum_{(i,j)\in\mathcal{J}(M,k)} v^s_{2,i}\,\partial_y v^s_{1,j}\right)\right)\\
&\quad + \frac{1}{\mu}\,\Pi\left(\partial_\theta^{-1}(h)\left(\sum_{(i,j)\in\mathcal{J}(M,k+2)} v^s_{1,i}\,\partial_\theta v^s_{1,j} + \sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{2,i}\,\partial_y v^s_{1,j}\right)\right),
\end{aligned}$$

with

$$S^{l2}_{2,k} := \frac{1}{\mu^2}\,\partial_\theta^{-1}(h)\,\left(-h\,\partial_y + \lambda\partial_{\theta\theta}\right)\partial_{\theta\theta}^{-2}(I-\Pi)(\partial_t v^s_{1,k-1} - \mu\partial_{yy} v^s_{1,k-1} - \lambda\partial_{\theta y} v^s_{2,k-1})\Big)\ , \tag{56}$$

$$\begin{aligned}
P^{l2}_2 f :=& \frac{1}{\mu^2}\,\Pi\left(\partial_\theta^{-1}(h)\,(-h\,\partial_y + \lambda\partial_{\theta\theta})\,\partial_{\theta\theta}^{-2}\,(I-\Pi)\,(h\partial_y f - \lambda\partial_{\theta\theta} f)\right)\\
&- \frac{1}{\mu}\,\Pi\left(\partial_\theta^{-1}(h)(\partial_t f - \mu\partial_{yy} f)\right)\ ,
\end{aligned} \tag{57}$$

$$Q^{l2}_2 f := \frac{\lambda}{\mu}\,\Pi\left(\partial_\theta^{-1}(h)(\partial_{\theta y} f)\right)\ . \tag{58}$$

Briefly, for all $k\in\mathbb{N}$, we denote

$$S^l_{2,k} := S^{l1}_{2,k} + S^{l2}_{2,k} + \lambda\,\partial_{yy}\Pi v^s_{2,k-1}\,, \qquad P^l_2 := P^{l1}_2 + P^{l2}_2\,, \qquad Q^l_2 := Q^{l1}_2 + Q^{l2}_2\,.$$

∘ Conclusion. Combining all informations together, we finally obtain (50) with

$$\begin{aligned}
S^{nl/\!/}_{1,k} :=& -\frac{1}{\mu}\,\Pi\left(\partial_{\theta\theta}^{-2}(h)\ \partial_y\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{1,i}\,\partial_\theta v^s_{1,j} + \sum_{(i,j)\in\mathcal{J}(M,k)} v^s_{2,i}\,\partial_y v^s_{1,j}\right)\right)\\
&- \Pi\left(\sum_{(i,j)\in\mathcal{J}(M,k+2)} v^s_{1,i}\,\partial_\theta v^s_{1,j} + \sum_{(i,j)\in\mathcal{J}(M,k+1)} v^s_{2,i}\,\partial_y v^s_{1,j}\right),
\end{aligned} \tag{59}$$

and

$$
\begin{aligned}
S_{2,k}^{nl/\!/} &= -\frac{1}{\mu}\,\Pi\left(\partial_{\theta\theta}^{-2}(h)\;\partial_y\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v_{1,i}^s\,\partial_\theta v_{1,j}^s+\sum_{(i,j)\in\mathcal{J}(M,k)} v_{2,i}^s\,\partial_y v_{1,j}^s\right)\right)\\
&-\frac{1}{\mu^2}\,\Pi\left(\partial_\theta h\;\partial_{\theta\theta}^{-2}(h)\;\partial_y\partial_{\theta\theta}^{-2}(I-\Pi)\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v_{1,i}^s\,\partial_\theta v_{1,j}^s+\sum_{(i,j)\in\mathcal{J}(M,k)} v_{2,i}^s\,\partial_y v_{1,j}^s\right)\right)\\
&-\frac{1}{\mu^2}\,\Pi\left(\partial_\theta^{-1}(h)\;(h\,\partial_y-\lambda\partial_{\theta\theta})\partial_{\theta\theta}^{-2}(I-\Pi)\left(\sum_{(i,j)\in\mathcal{J}(M,k+1)} v_{1,i}^s\,\partial_\theta v_{1,j}^s+\sum_{(i,j)\in\mathcal{J}(M,k)} v_{2,i}^s\,\partial_y v_{1,j}^s\right)\right)\\
&-\frac{1}{\mu}\,\Pi\left(\partial_\theta^{-1}(h)\left(\sum_{(i,j)\in\mathcal{J}(M,k+2)} v_{1,i}^s\,\partial_\theta v_{1,j}^s+\sum_{(i,j)\in\mathcal{J}(M,k+1)} v_{2,i}^s\,\partial_y v_{1,j}^s\right)\right)\\
&-\Pi\left(\sum_{(i,j)\in\mathcal{J}(M,k+2)} v_{1,i}^s\,\partial_\theta v_{2,j}^s+\sum_{(i,j)\in\mathcal{J}(M,k+1)} v_{2,i}^s\,\partial_y v_{2,j}^s\right).
\end{aligned}
\tag{60}
$$

□

The definition of $S_k^{nl/\!/}$ involves the sets $\mathcal{J}(M,k)$, $\mathcal{J}(M,k+1)$ and $\mathcal{J}(M,k+2)$. Note that

$$\big[\,M\geq 3\,,\quad \tilde{k}\in\{k,k+1,k+2\}\,,\quad (i,j)\in\mathcal{J}(M,\tilde{k})\,\big]\quad\Longrightarrow\quad i\leq k-1 \text{ and } j\leq k-1\,. \tag{61}$$

Thus, the presence of v_k^s inside $S_k^{nl/\!/}$ is not allowed as long as $M\geq 3$. On the contrary, when $M=2$, it becomes effective. This remark can be formalized through the following statement.

Lemma 2.4. *[Refined description of the source term of (50)]*

i) The expressions $S_k^{l/\!/}$ only depend on the v_j^s with $j\in[\![0,k-1]\!]$. More precisely, we have

$$S_k^{l/\!/}=f_k^{l/\!/}(v_0^s,\dots,v_{k-1}^s)$$

where $f_k^{l/\!/}:={}^t(f_{1,k}^{l/\!/},f_{2,k}^{l/\!/})$ are homogeneous linear functions of their arguments.

ii) When $M\geq 3$, the expressions $S_k^{nl/\!/}$ only depend on the v_j^s with $j\in[\![0,k-1]\!]$. More precisely, we can retain that

$$S_k^{nl/\!/}=f_k^{qnl/\!/}(v_0^s,\dots,v_{k-1}^s)$$

where $f_k^{qnl/\!/}:={}^t(f_{1,k}^{qnl/\!/},f_{2,k}^{qnl/\!/})$ are quadratic functions of their arguments.

iii) When $M=2$, the expressions $S_k^{nl/\!/}$ only depend on the v_j^s with $j\in[\![0,k]\!]$. The influence of v_k^s can be specified through a decomposition of the form

$$S_k^{nl/\!/}=f_k^{qnl/\!/}(v_0^s,\dots,v_{k-1}^s)+SP(v_0^s)\,v_k^s\,,\qquad f_k^{qnl/\!/}:=(f_{1,k}^{qnl/\!/},f_{2,k}^{qnl/\!/})\,.$$

Again $f_k^{qnl/\!/}$ is a quadratic function of its arguments whereas $SP(v_0^s):={}^t\big(SP_1(v_0^s),SP_2(v_0^s)\big)$ is the linear differential operator defined according to

$$
\left\{
\begin{aligned}
&SP_1(v_0^s)\,v_k^s:=-\,\Pi\,(v_{1,k}^s\,\partial_\theta v_{1,0}^s+v_{1,0}^s\,\partial_\theta v_{1,k}^s)\,,\\
&SP_2(v_0^s)\,v_k^s:=-\,\Pi(v_{1,0}^s\,\partial_\theta v_{2,k}^s+v_{1,k}^s\,\partial_\theta v_{2,0}^s)-\frac{1}{\mu}\,\Pi\,\big(\partial_\theta^{-1}(h)\,(v_{1,0}^s\,\partial_\theta v_{1,k}^s+v_{1,k}^s\,\partial_\theta v_{1,0}^s)\big)\,.
\end{aligned}
\right.
\tag{62}
$$

From now on, we simply note $f_k^{nl/\!/} := f_k^{l/\!/} + f_k^{qnl/\!/}$.

Proof of Lemma 2.4. As already mentioned, the statement *i)* is a direct consequence of (61). The *linear* aspect of $f_k^{l/\!/}$ is a consequence of the formulas obtained in (53) and (56). On the other hand, the *quadratic* aspect of $f_k^{qnl/\!/}$ is obvious in view of (59) and (60). It proves *ii)*.

There remains to consider the situation *iii)* where $M = 2$. Note that $\mathcal{J}(2, k+2) = \mathcal{J}(0, k)$. In view of (61), we have to concentrate on the contribution of

$$\begin{pmatrix} -\Pi\Big(\displaystyle\sum_{(i,j)\in\mathcal{J}(0,k)} v_{1,i}^s\,\partial_\theta v_{1,j}^s\Big) \\ -\dfrac{1}{\mu}\Pi\left(\partial_\theta^{-1}(h)\Big(\displaystyle\sum_{(i,j)\in\mathcal{J}(0,k)} v_{1,i}^s\,\partial_\theta v_{1,j}^s\Big)\right) - \Pi\Big(\displaystyle\sum_{(i,j)\in\mathcal{J}(0,k)} v_{1,i}^s\,\partial_\theta v_{2,j}^s\Big) \end{pmatrix}.$$

In the sums above, only the extremal indices $(i,j) = (k,0)$ and $(i,j) = (0,k)$ give a contribution to include in $SP(v_0^s)$, leading to (62). $\square$

2.1.2 Analysis of the system (50)

Projection of system (50)**.** The system (50) is clearly an evolution equation of parabolic type. As such, it can be completed by initial data. But to solve it, we also need to identify the extra terms $(I-\Pi)v_{\star,k}^s$ with $\star \in \{1,2\}$. To this end, it suffices again to exploit (42). Recall that $\mathcal{V} \in \{W^{m,p}, H^s, \mathcal{W}_T^{m,s}, \mathcal{H}_T^s, \mathcal{E}_\delta^s\}$ and define the linear continuous isomorphism

$$\begin{array}{rcl} \Phi : \mathcal{V}(\mathbb{T}\times\mathbb{R}) \times \mathcal{V}(\mathbb{T}\times\mathbb{R}) & \longrightarrow & \mathcal{V}^{tot} := \mathcal{V}^\perp(\mathbb{T}\times\mathbb{R}) \times \mathcal{V}^{/\!/}(\mathbb{R}) \times \mathcal{V}^\perp(\mathbb{T}\times\mathbb{R}) \times \mathcal{V}^{/\!/}(\mathbb{R}) \\ (f_1, f_2) & \longmapsto & \big((I-\Pi)f_1, \Pi f_1, (I-\Pi)f_2, \Pi f_2\big)\,. \end{array}$$

By construction, the expression $V_k^s := {}^t\Phi\, v_k^s$ must be solution of the system

$$\mathcal{A}\,V_k^s := \begin{pmatrix} \mu\partial_{\theta\theta} & 0 & 0 & 0 \\ -P_1^{l\perp} & P_y & 0 & 0 \\ -T_s & -T_s & \mu\partial_{\theta\theta} & 0 \\ -P_2^{l\perp} & -P_2^{l/\!/} & -Q_2^{l\perp} & P_y \end{pmatrix} V_k^s = \begin{pmatrix} f_{1,k}^{nl\perp} \\ f_{1,k}^{nl/\!/} \\ f_{2,k}^{nl\perp} \\ f_{2,k}^{nl/\!/} \end{pmatrix} =: f_k^{nl}, \qquad V_k^s = \begin{pmatrix} v_{1,k}^{s\perp} \\ v_{1,k}^{s/\!/} \\ v_{2,k}^{s\perp} \\ v_{2,k}^{s/\!/} \end{pmatrix} \tag{63}$$

where the term $f_k^{nl/\!/}$ is given by Lemma 2.4 whereas T_s and P_y are the operators defined by

$$T_s f := (I-\Pi)\,(\partial_\theta h\, f)\,, \qquad P_y f := \partial_t f - \Big(\mu + \frac{1}{\mu}\Pi\big((\partial_\theta^{-1}h)^2\big)\Big)\,\partial_{yy} f.$$

Description of $f_k^{nl\perp}$. As for $f_k^{nl/\!/}$ we decompose $f_k^{nl\perp}$ into $f_k^{nl\perp} := f_k^{l\perp} + f_k^{qnl\perp}$ where the linear part $f_k^{l\perp}$ is defined by

$$f_{1,k}^{l\perp} := (I-\Pi)\left(\partial_t v_{1,k-2}^s - \mu\partial_{yy}v_{1,k-2}^s - \lambda\partial_{\theta y}v_{2,k-2}^s + h\,\partial_y v_{1,k-1}^s - \lambda\partial_{\theta\theta}v_{1,k-1}^s\right), \tag{64}$$
$$f_{2,k}^{l\perp} := (I-\Pi)(\partial_t v_{2,k-2}^s - \mu\partial_{yy}v_{2,k-2}^s - \lambda\partial_{\theta y}v_{1,k-2}^s + h\,\partial_y v_{2,k-1}^s - \lambda\partial_{yy}v_{2,k-3}^s)\,, \tag{65}$$

whereas the quadratic part $f_k^{qnl\perp} = {}^t(f_{1,k}^{qnl\perp}, f_{2,k}^{qnl\perp})$ is given by

$$f_{p,k}^{qnl\perp} := (I-\Pi)\Bigg(\sum_{(i,j)\in\mathcal{J}(M,k)} v_{1,i}^s\,\partial_\theta v_{p,j}^s + \sum_{(i,j)\in\mathcal{J}(M+1,k)} v_{2,i}^s\,\partial_y v_{p,j}^s\Bigg)\,, \qquad p \in \{1,2\}\,. \tag{66}$$

Lemma 2.5. *[Description of $f_k^{nl\perp}$] The functions $f_k^{l\perp}$ and $f_k^{qnl\perp}$ only depend on the v_j^s with $j \in [\![0, k-1]\!]$. They are respectively homogeneous linear and homogeneous quadratic functions of their arguments $(v_0^s, \dots, v_{k-1}^s)$.*

Proof of Lemma 2.5. Just look at (64)-(65)-(66) together with (61). $\square$

Analysis of the system (63). To study the system (63), we have to distinguish the general case $M \geq 3$ from the critical case $M = 2$.

∘ When $M \geq 3$, According to the Lemmas 2.3 and 2.5, the above right hand term f_k^{nl} can be viewed as a source term.

∘ When $M = 2$, $f_k^{nl/\!/}$ is no longer a source term. Now, recall that we can exploit the condition (48). This information is essential. It induces many simplifications when computing $SP(v_0^s)$. We find that $SP_1(v_0^s) \equiv 0$ whereas $SP_2(v_0^s)$ can be reduced to the following linear (non differential) operator

$$SP_2(v_0^s)\, v_k^s = -\,\Pi(v_{1,k}^{s\perp}\,\partial_\theta v_{2,0}^s) + \frac{1}{\mu}\; v_{1,0}^{s/\!/}\;\Pi\,(h\, v_{1,k}^{s\perp})\,. \tag{67}$$

At first sight, the expression $SP_2(v_0^s)\, v_0^s$ depends in a non linear way on v_0^s. However, we can again exploit the condition (48) (which says that $v_{1,0}^{s\perp} \equiv 0$) and then apply (67) with $k = 0$ in order to obtain further cancellations. There remains

$$SP(v_0^s)\, v_0^s \;\equiv\; 0\,. \tag{68}$$

From now on, since there is no more ambiguity, we can omit to signal that $SP(v_0^s)$ depends on v_0^s, and in fact only on $v_{1,0}^{s/\!/}$. We will most often note $SP(v_0^s) \equiv SP(v_0^{s/\!/}) \equiv SP = {}^t(SP_1, SP_2)$. Since $SP_2 \not\equiv 0$, the formulation (63) must be changed. This time, we have to deal with

$$\widetilde{\mathcal{A}}\, V_k^s = {}^t(f_{1,k}^{nl\perp}, f_{1,k}^{nl/\!/}, f_{2,k}^{nl\perp}, f_{2,k}^{nl/\!/})\,, \qquad V_k^s := {}^t\Phi\, v_k^s \tag{69}$$

where

$$\widetilde{\mathcal{A}} := \begin{pmatrix} \mu\partial_{\theta\theta} & 0 & 0 & 0 \\ -P_1^{l\perp} & P_y & 0 & 0 \\ -T_s & -T_s & \mu\partial_{\theta\theta} & 0 \\ -P_2^{l\perp} - SP_2 & -P_2^{l\perp} & -Q_2^{l\perp} & P_y \end{pmatrix}. \tag{70}$$

The similarities between $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are obvious. These two matrix valued operators have both a triangular structure. The difference, when passing from $\mathcal{A}$ to $\widetilde{\mathcal{A}}$, concerns only the perturbation in the bottom-left position $(4, 1)$. This particularity plays a crucial part in the discussion below. To simplify, we present below the result in a smooth setting.

Lemma 2.6. *[Solving the system (63) or (69)] We assume that the condition (48) is verified. Select a function $V_0^{/\!/} = {}^t(V_0^{1/\!/}, V_0^{2/\!/}) \in H^\infty(\mathbb{R})^2$ and a source term $F = {}^t(F^{1\perp}, F^{1/\!/}, F^{2\perp}, F^{2/\!/}) \in (\mathcal{H}_T^\infty)^{tot}$. Then, for all $T \in \mathbb{R}_+^*$, the problem*

$$\{\, \mathcal{P}\, V = F\,, \qquad V = {}^t(V^{1\perp}, V^{1/\!/}, V^{2\perp}, V^{2/\!/})\,, \qquad {}^t(V^{1/\!/}, V^{2/\!/})_{|t=0} = {}^t(V_0^{1/\!/}, V_0^{2/\!/}), \tag{71}$$

where $\mathcal{P} \in \{\mathcal{A}, \widetilde{\mathcal{A}}\}$, has a unique solution V in $(\mathcal{H}_T^\infty)^{tot}$.

Proof of Lemma 2.6. To solve (71), the strategy is to argue line after line. We start to solve the problem for the operator $\mathcal{A}$.

- *First line.* Since the operator $\mu\partial_{\theta\theta} : (\mathcal{H}_T^\infty)^\perp \longrightarrow (\mathcal{H}_T^\infty)^\perp$ is invertible, we can define without ambiguity (and with no choice)

$$V^{1\perp} := \mu^{-1}\partial_{\theta\theta}^{-2} F^{1\perp} \in (\mathcal{H}_T^\infty)^\perp \ . \tag{72}$$

- *Second line.* Observe that $P_1^{l\perp} : \mathcal{H}_T^\infty \longrightarrow (\mathcal{H}_T^\infty)^\perp$. The next component $V^{1/\!/}$ must be a solution of the heat equation (in t and y)

$$\left\{ \, P_y V^{1/\!/} = F^{1/\!/} + P_1^{l\perp} V^{1\perp} \in (\mathcal{H}_T^\infty)^{/\!/} \, , \qquad (V^{1/\!/})_{|t=0} = V_0^{1/\!/} \in (\mathcal{H}_T^\infty)^{/\!/} \, . \right. \tag{73}$$

Obviously, there is a unique solution on $[0,T]$ of this initial value problem. It does not depend on θ. In other words, it is such that $V(t,\cdot) \in (\mathcal{H}_T^\infty)^{/\!/}$ for all $t \in [0,T]$.

- *Third line.* Since $T_s : \mathcal{H}_T^\infty \longrightarrow (\mathcal{H}_T^\infty)^\perp$ the component $V^{2\perp}$ can be obtained through the formula

$$V^{2\perp} = \mu^{-1}\partial_{\theta\theta}^{-2}\big(F^{2\perp} + T_s V^{1\perp} + T_s V^{1/\!/}\big) \in (\mathcal{H}_T^\infty)^\perp \ . \tag{74}$$

- *Fourth line.* We can use the same argument as in the second line. It suffices to check that by definition $P_2^l : \mathcal{H}_T^\infty \longrightarrow (\mathcal{H}_T^\infty)^\perp$ and $Q_2^l : \mathcal{H}_T^\infty \longrightarrow (\mathcal{H}_T^\infty)^\perp$. Then, there remains to solve

$$\left\{ \, P_y V^{2/\!/} = F^{2/\!/} + P_2^{l\perp} V^{1\perp} + P_2^{l/\!/} V^{1/\!/} + Q_2^{l\perp} V^{2\perp} \in (\mathcal{H}_T^\infty)^{/\!/} \, , \qquad (V^{2/\!/})_{|t=0} = V_0^{2/\!/} \in (\mathcal{H}_T^\infty)^{/\!/} \, . \right.$$

Note that the triangular structure of $\mathcal{A}$ is crucial in this procedure.

To solve the system (69) that is replacing the operator $\mathcal{A}$ by $\widetilde{\mathcal{A}}$, the only change in the above proof is at the level of the fourth line where the supplementary source term $SP_2V^{1\perp}$ must be incorporated. ☐

2.1.3 Analysis of the system (43)

Projection of the system (43). In this paragraph, we consider the parabolic system (43) which can be associated with some smooth initial data $v_k^f(0,\cdot) \in \mathcal{H}^\infty(\mathbb{T}\times\mathbb{R})$. Classical statements (see for instance [11]) say that the corresponding Cauchy problem has a unique global solution v_k^f such that $v_k^f(t,\cdot) \in \mathcal{H}(\mathbb{T}\times\mathbb{R})$ for all $\tau \in \mathbb{R}_+$. The difficulty is the following. The variable τ is aimed to be replaced by $\varepsilon^{-2}\, t$ with t fixed and $\varepsilon \to 0$ and, since the original equation (3) contains nonlinearities, we cannot allow any (uncontrolled) growth with respect to τ. To get round this problem, we will instead require a rapid decay when $\tau \to +\infty$ but this necessitates $v_k^f(0,\cdot)$ to be selected conveniently.

To see how to adjust $v_k^f(0,\cdot)$, we can interpret (43) in the form

$$\mathcal{B}\, V_k^f := \begin{pmatrix} P_\theta & 0 & 0 & 0 \\ 0 & \partial_\tau & 0 & 0 \\ T_s & T_s & P_\theta & 0 \\ T_f & 0 & 0 & \partial_\tau \end{pmatrix} \begin{pmatrix} g_{1,k}^{nl\perp} \\ g_{1,k}^{nl/\!/} \\ g_{2,k}^{nl\perp} \\ g_{2,k}^{nl/\!/} \end{pmatrix} , \qquad V_k^f := {}^t\Phi(v_k^f) = \begin{pmatrix} v_{1,k}^{f\perp} \\ v_{1,k}^{f/\!/} \\ v_{2,k}^{f\perp} \\ v_{2,k}^{f/\!/} \end{pmatrix} \tag{75}$$

where T_f and P_θ are the operators defined by

$$T_f f := \Pi(\partial_\theta h\, f) \, , \qquad P_\theta f := \partial_\tau f - \mu\, \partial_{\theta\theta} f \, .$$

Description of $g_k^{nl\perp}$. We decompose $g_k^{nl\perp}$ into is linear and quadratic part $g_k^{nl} := g_k^l + g_k^{qnl}$. The linear part g_k^l is defined by

$$\begin{cases} g_{1,k}^l := (-h\,\partial_y v_{1,k-1}^f + \lambda \partial_{\theta\theta} v_{1,k-1}^f) + (\mu \partial_{yy} v_{1,k-1}^f + \lambda \partial_{\theta y} v_{2,k-2}^f)\,, \\ g_{2,k}^l := -h\,\partial_y v_{2,k-1}^f + \Big(\lambda \partial_{\theta y} v_{1,k-2}^f + \mu \partial_{yy} v_{2,k-2}^f\Big) + \lambda \partial_{yy} v_{2,k-3}^f\,, \end{cases} \tag{76}$$

whereas, the quadratic part of $g_k^{qnl} = {}^t(g_{1,k}^{qnl}, g_{2,k}^{qnl})$ is given by

$$\begin{aligned} g_{1,k}^{qnl} := & \sum_{(i,j)\in \mathcal{J}(M,k)} v_{1,i}^f\,\partial_\theta v_{1,j}^f + \sum_{(i,j)\in \mathcal{J}(M,k-1)} v_{2,i}^f\,\partial_y v_{1,j}^f \\ & + \sum_{(i,j,l)\in \mathcal{I}(M,k)} \Big(v_{1,i}^f\,(\partial_t)^l\,(\partial_\theta v_{1,j}^s)(0) + (\partial_t)^l\,(v_{1,i}^s)(0)\,\partial_\theta v_{1,j}^f \Big)\,\frac{\tau^l}{l!} \\ & + \sum_{(i,j,l)\in \mathcal{I}(M,k-1)} \Big(v_{2,i}^f\,(\partial_t)^l\,(\partial_y v_{1,j}^s)(0) + (\partial_t)^l\,(v_{2,i}^s)(0)\,\partial_y v_{1,j}^f \Big)\,\frac{\tau^l}{l!}\,, \end{aligned}$$

$$\begin{aligned} g_{2,k}^{qnl} := & \sum_{(i,j)\in \mathcal{J}(M,k)} v_{1,i}^f\,\partial_\theta v_{2,j}^f + \sum_{(i,j)\in \mathcal{J}(M,k-1)} v_{2,i}^f\,\partial_\theta v_{2,j}^f \\ & + \sum_{(i,j,l)\in \mathcal{I}(M,k)} \Big(v_{1,i}^f\,(\partial_t)^l\,(\partial_\theta v_{2,j}^s)(0) + (\partial_t)^l\,(v_{1,i}^s)(0)\,\partial_\theta v_{2,j}^f \Big)\,\frac{\tau^l}{l!} \\ & + \sum_{(i,j,l)\in \mathcal{I}(M,k-1)} \Big(v_{2,i}^f\,(\partial_t)^l\,(\partial_y v_{2,j}^s)(0) + (\partial_t)^l\,(v_{2,i}^s)(0)\,\partial_y v_{2,j}^f \Big)\,\frac{\tau^l}{l!}\,. \end{aligned}$$

Lemma 2.7. *[Description of g_k^{nl}] Assume that $M \geq 2$. The functions g_k^l and g_k^{qnl} depend only on the v_j^f and the $\partial_t^l v_j^s(0,\cdot)$ where $j \in [\![0, k-1]\!]$ and $l \in [\![0, \lfloor \frac{k}{2} \rfloor]\!]$. They are respectively homogeneous linear and homogeneous quadratic functions of their arguments.*

Proof of Lemma 2.7. It suffices to examine the various terms appearing in the sums involved by the definition of g_k^{qnl}.

- The sums based on the symbol $\mathcal{J}$ can be dealt by observing that

$$\big[\, M \geq 2\,,\quad \tilde{k} \in \{k-1, k\}\,,\quad (i,j) \in \mathcal{J}(M,\tilde{k})\,\big] \quad \Longrightarrow \quad i \leq k-1 \text{ and } j \leq k-1\,. \tag{77}$$

- The sums involving the symbol $\mathcal{I}$ are of the form $\mathcal{I}(M,k)$ or $\mathcal{I}(M,k-1)$. Coming back to the definition (41), we can easily infer that

$$\big[\, M \geq 2\,,\quad \tilde{k} \in \{k-1, k\}\,,\quad (i,j,l) \in \mathcal{I}(M,\tilde{k})\,\big] \quad \Longrightarrow \quad i \leq k-1 \text{ and } j \leq k-1 \tag{78}$$

  as well as $l \leq \lfloor \frac{k}{2} \rfloor$.

$\square$

Due to Lemma 2.7, the expression g_k^{nl} can be viewed as a source term in system (75). In order to guarantee the fast decaying criterion in τ, we can proceed as described below.

Lemma 2.8. *[Solving 75 in the case of a fast decay when τ tends to $+\infty$] Select functions*

$$V_0^\perp = {}^t(V_0^{1\perp}, V_0^{2\perp}) \in H^\infty(\mathbb{T}\times\mathbb{R})^2 \,, \qquad G = {}^t(G^{1\perp}, G^{1/\!/}, G^{2\perp}, G^{2/\!/}) \in (\mathcal{E}_\delta^\infty)^{tot} \,, \qquad \delta \in]0, \mu[\,.$$

There is a unique expression $V_0^{/\!/} = {}^t(V_0^{1/\!/}, V_0^{2/\!/}) \in (H^\infty)^{/\!/}(\mathbb{T}\times\mathbb{R};\mathbb{R})^2$ which can be determined in function of G through formulas (80) and (81) such that the Cauchy problem

$$\left\{ \mathcal{B}\, V = G \,, \qquad V = {}^t(V^{1\perp}, V^{1/\!/}, V^{2\perp}, V^{2/\!/}) \,, \qquad V_{|\tau=0} = {}^t(V_0^{1\perp}, V_0^{1/\!/}, V_0^{2\perp}, V_0^{2/\!/}) \right. \tag{79}$$

has a global solution V belonging to the space $(\mathcal{E}_\delta^\infty)^{tot}$.

Proof of Lemma 2.8. The strategy is again to argue line after line.

- *First line.* Just apply the end of Lemma 2.2.

- *Second line.* It suffices to take

$$V_0^{1/\!/}(.) := -\int_0^{+\infty} G^{1/\!/}(s,.)\; ds \in (H^\infty)^{/\!/} \tag{80}$$

in order to recover after integration that $V^{1/\!/} \in (\mathcal{E}_\delta^\infty)^{/\!/}$ with

$$V^{1/\!/}(\tau,.) = V_0^{1/\!/}(.) + \int_0^\tau G^{1/\!/}(s,.)\; ds = -\int_\tau^{+\infty} G^{1/\!/}(s,.)\; ds \,.$$

- *Third line.* For all $m\in\mathbb{N}$, the operator $T_s : H^m \to (H^m)^\perp$ is continuous. It follows that T_s sends the functional space $\mathcal{E}_\delta^\infty$ into $(\mathcal{E}_\delta^\infty)^\perp$. Concerning $V^{2\perp}$, the argument is again Lemma 2.2 applied this time with the source term $G^{2\perp} - T_s V^{1\perp} - T_s V^{1/\!/} \in (\mathcal{E}_\delta^\infty)^\perp$.

- *Fourth line.* For all $m\in\mathbb{N}$, the operator $T_f : H^m \to (H^m)^{/\!/}$ is continuous. Therefore, we know that $T_f : \mathcal{E}_\delta^\infty \to (\mathcal{E}_\delta^\infty)^{/\!/}$. With this in mind, it suffices to select

$$V_0^{2/\!/}(.) := -\int_0^{+\infty} (G^{2/\!/} - T_f G^{1\perp})(s,.)\; ds \in (H^\infty)^{/\!/} \,. \tag{81}$$

$\square$

2.1.4 Proof of Proposition 2.1

The matter is to show by induction on $K \in [\![0, N+1]\!]$ that the property given at the level of line (105) is verified.

Verification of $\mathcal{HN}(0)$. By convention, we start with $v_k^s \equiv 0$ and $v_k^f \equiv 0$ for $k \in \{-3,-2,-1\}$. Applying Lemmas 2.4, 2.5 and 2.7 with $k=0$ and exploiting the given (linear or quadratic) homogeneity properties, we find that $f_0^{nl} \equiv 0$ and $g_0^{nl} \equiv 0$.

Recall that $V_0^s := {}^t\Phi\, v_0^s$ and $V_0^f := {}^t\Phi\, v_0^f$. The matter here is to show the existence of functions $V_0^s \in (\mathcal{H}_T^\infty)^{tot}$ and $V_0^f \in (\mathcal{E}_\delta^\infty)^{tot}$ such that:

- For $M \geq 3$:

$$\mathcal{A}\, V_0^s = 0 \,, \qquad \mathcal{B}\, V_0^f = 0 \,, \qquad (V_0^s + V_0^f)(0,\cdot) = {}^t\Phi\, v_0^0(\cdot) \,. \tag{82}$$

- For $M = 2$:
$$\widetilde{\mathcal{A}}\, V_0^s = 0\,, \qquad \mathcal{B}\, V_0^f = 0\,, \qquad (V_0^s + V_0^f)(0,\cdot) = {}^t\Phi\, v_0^0(\cdot)\,. \tag{83}$$
By construction, the two first lines of (82) and (83) amount to the same thing as $\mu\, \partial_{\theta\theta} v_{1,0}^{s\perp} \equiv 0$. We recover here (48). From (68), we can deduce that $\widetilde{\mathcal{A}}\, V_0^s \equiv \mathcal{A}\, V_0^s$. Therefore, the discussion concerning (83) is the same as the one related to (82).

The above initial condition can be decomposed into
$$ {}^t(V_0^{s1\perp}, V_0^{s2\perp})(0,\cdot) + {}^t(V_0^{f1\perp}, V_0^{f2\perp})(0,\cdot) = (I - \Pi) v_0^0(\cdot)\,, \tag{84}$$
$$ {}^t(V_0^{s1/\!/}, V_0^{s2/\!/})(0,\cdot) + {}^t(V_0^{f1/\!/}, V_0^{f2/\!/})(0,\cdot) = \Pi v_0^0(\cdot)\,. \tag{85}$$
In view of (80) and (81), we must have $V_0^{f1/\!/}(0,\cdot) \equiv V_0^{f2/\!/}(0,\cdot) \equiv 0$ whatever $V_0^{f\perp}(0,\cdot)$ is. It follows that we can identify $V_0^{s1/\!/}(0,\cdot)$ and $V_0^{s2/\!/}(0,\cdot)$ through (85). Now, knowing what is $V_0^{s1/\!/}(0,\cdot)$, formulas (72) and (74) give access to $V_0^{s1\perp}(0,\cdot)$ and $V_0^{s2\perp}(0,\cdot)$. There remains to use the condition (84) in order to further extract $V_0^{f1\perp}(0,\cdot)$ and $V_0^{f2\perp}(0,\cdot)$.

We apply Lemma 2.6 and Lemma 2.8 in the case of the initial data $V_0^{s/\!/}(0,\cdot)$ and $V_0^{f\perp}(0,\cdot)$ which have just been computed. Note that, due to the preceding construction, there is no contradiction between the expressions $V_0^{s\perp}(0,\cdot)$ and $V_0^{f/\!/}(0,\cdot)$ thus obtained and the compatibility conditions required at the level of (84) and (85). By this way, we can recover functions $V_0^s \in (\mathcal{H}_T^\infty)^{tot}$ and $V_0^f \in (\mathcal{E}_\delta^\infty)^{tot}$. Then, to conclude, it suffices to come back to $v_0^s \in \mathcal{H}_T^\infty$ and $v_0^f \in \mathcal{E}_\delta^\infty$ through the action of Φ^{-1}.

Assume that the condition $\mathcal{HN}(K)$ is true for some $K \in [\![0, N]\!]$. Since the criterion (48) is satisfied, the problem can be interpreted as before. The matter is to find two functions $V_{K+1}^s := {}^t\Phi\, v_k^s \in (\mathcal{H}_T^\infty)^{tot}$ and $V_{K+1}^f := {}^t\Phi\, v_k^f \in (\mathcal{E}_\delta^\infty)^{tot}$ such that:

- For $M \geq 3$:
$$\mathcal{A}\, V_{K+1}^s = {}^t\Phi\, f_k^{nl}, \qquad \mathcal{B}\, V_{K+1}^f = {}^t\Phi\, g_k^{nl}, \qquad (V_{K+1}^s + V_{K+1}^f)(0,\cdot) = {}^t\Phi\, v_{K+1}^0(\cdot)\,.$$
- For $M = 2$:
$$\widetilde{\mathcal{A}}\, V_{K+1}^s = {}^t\Phi f_k^{nl}\,, \qquad \mathcal{B}\, V_{K+1}^f = {}^t\Phi\, g_k^{nl}, \qquad (V_{K+1}^s + V_{K+1}^f\ )(0,\cdot) = {}^t\Phi\, v_{K+1}^0(\cdot)\,.$$
The induction hypothesis applied with the index K together with Lemmas 2.4, 2.5 and 2.7 say that the functions ${}^t\Phi\, f_k^{nl}$ and ${}^t\Phi\, g_k^{nl}$ are known source terms with the expected $(\mathcal{H}_T^\infty)^{tot}$ and $(\mathcal{E}_\delta^\infty)^{tot}$ regularities.

-When $M \geq 3$, the initial condition can be decomposed into
$$ {}^t(V_{K+1}^{s1\perp}, V_{K+1}^{s2\perp})(0,\cdot) + {}^t(V_{K+1}^{f1\perp}, V_{K+1}^{f2\perp})(0,\cdot) = (I - \Pi) v_{K+1}^0(\cdot)\,, \tag{86}$$
$$ {}^t(V_{K+1}^{s1/\!/}, V_{K+1}^{s2/\!/})(0,\cdot) + {}^t(V_{K+1}^{f1/\!/}, V_{K+1}^{f2/\!/})(0,\cdot) = \Pi v_{K+1}^0(\cdot)\,. \tag{87}$$
In view of (80) and (81), as clearly indicated in the statement of Lemma 2.8, the expression $V_{K+1}^{f/\!/}(0,\cdot)$ depends only on ${}^t\Phi g_{K+1}^{nl}$. We can determine $V_{K+1}^{s/\!/}(0,\cdot)$ through (87). Knowing what is $V_{K+1}^{s/\!/}(0,\cdot)$, formulas (72) and (74) give access to $V_{K+1}^{s\perp}(0,\cdot)$. There remains to use the condition

(86) in order to deduce $V^{f1\perp}_{K+1}(0,\cdot)$ and $V^{f2\perp}_{K+1}(0,\cdot)$. Remark that the initial data $V^{s}_{K+1}(0,\cdot)$ and $V^{f}_{K+1}(0,\cdot)$ thus obtained inherit the expected $\mathcal{H}^\infty(\mathbb{T}\times\mathbb{R})$ smoothness.

Again, we apply Lemmas 2.6 and 2.8 in the case of the initial data $V^{s/\!/}_{K+1}(0,\cdot)$ and $V^{f\perp}_{K+1}(0,\cdot)$ which have just been computed. As before, the preceding choices concerning $V^{f/\!/}_{K+1}(0,\cdot)$ and $V^{s\perp}_{K+1}(0,\cdot)$ are sufficient to guarantee (87) and (86). We find that $V^{s}_{K+1}\in(\mathcal{H}^\infty_T)^{tot}$ and $V^{f}_{K+1}\in(\mathcal{E}^\infty_\delta)^{tot}$. To conclude, it suffices to come back to $v^s_{K+1}\in\mathcal{H}^\infty_T$ and $v^f_{K+1}\in\mathcal{E}^\infty_\delta$ through Φ^{-1}.

-When $M=2$, the same types of arguments prevail. This ends the induction. $\square$

From the preceding construction, we can also deduce the following information.

Corollary 2.9. *[Nonlinear homogenization] For all $k\in[\![0,N+1]\!]$, the expression Πv^s_k can be determined through the following parabolic equation,*

$$\partial_t \Pi v^s_k - \left(\mu + \frac{1}{\mu}\Pi\left((\partial_\theta^{-1}h)^2\right)\right)\partial_{yy}\Pi v^s_k = S^{nl}_k\,, \tag{88}$$

where the source term $S^{nl}_k := {}^t(S^{nl}_{1,k},S^{nl}_{2,k})$ depends on the index j with $j<k$. This fact may be formulated by writing ${}^t(S^{nl}_{1,k},S^{nl}_{2,k}) = {}^t(f^{nl}_{1,k},f^{nl}_{2,k})(v^s_0,\dots,v^s_{k-1})$.

2.1.5 Approximated solutions

In this subsection, we prove estimate (11). We assume that $m\geq 2$ so that $H^m(\mathbb{R}^2)$ is an algebra.

We explicitly compute the action of the operator $\mathcal{L}^a$ on the approximated solution v^a_ε built in the previous subsection and estimate the remainder $R_\varepsilon := \mathcal{L}^a(\varepsilon, v^a_\varepsilon)$ in H^m-norm.
To this ends, we justify that the slow profiles v^{as}_ε and the fast profile v^{af}_ε are good approximation of operator $\mathcal{L}^{as}$ and $\mathcal{L}^{af}$.
One aspect of the proof is to compute the difference between $\mathcal{L}^{af}$and $\mathcal{L}^{aft}$ given by some Taylor formula. To control it, we provide the following Lemma.

Lemma 2.10. *Let $m\geq 2$ be an integer and $\delta\in]0,\mu[$. Let $f\in\mathcal{E}^m_\delta(\mathbb{T}\times\mathbb{R})$, $g\in\mathcal{H}^{m,0}_T(\mathbb{T}\times\mathbb{R})$. On the strip $[0,T]$, consider the function $h^\varepsilon_{exp}(t,\cdot) := f\left(\varepsilon^{-2}t,\cdot\right)\int_0^t u^N g(u,\cdot)du$. Then the family $\left\{\varepsilon^{-(N+1)}h^\varepsilon_{exp}\right\}_\varepsilon$ is bounded in $\mathcal{H}^{m,0}_T(\mathbb{T}\times\mathbb{R})$,* i.e.*:*

$$\sup_{\varepsilon\in]0,1]}\ \sup_{t\in[0,T]}\left\|\varepsilon^{-(N+1)}h^\varepsilon_{exp}(t,\cdot)\right\|_{H^m(\mathbb{T}\times\mathbb{R})} < +\infty.$$

The proof is obvious using some Gagliardo-Niremberg's estimate.

Proof of Proposition 11. First, we decompose the action of $\mathcal{L}^a$ on v^a_ε into

$$\mathcal{L}^a(\varepsilon, v^a_\varepsilon(t,\cdot)) = \mathcal{L}^{as}(\varepsilon, v^{as}_\varepsilon(t,\cdot)) + \mathcal{L}^{af}(\varepsilon, v^{as}_\varepsilon(t,\cdot), v^{af}_\varepsilon(t/\varepsilon^2,\cdot)), \tag{89}$$

where the operators $\mathcal{L}^{as}$ and $\mathcal{L}^{af}$ are defined in (32) and (36).

∘ Writing v_ε^{as} as the sum $v_\varepsilon^{as} = \sum_{k=0}^{N+1} \varepsilon^k v_k^s$ and using the cascade of equations (42), we obtain:

$$
\begin{aligned}
\mathcal{L}^{as}(\varepsilon, v_\varepsilon^{as}) := &\sum_{k=N}^{N+1} \varepsilon^k \partial_t v_k^s + \varepsilon^N \partial_y v_{N+1}^s \\
&+ \sum_{k=N}^{2(N+1)+(M-2)} \varepsilon^k \sum_{(i,j)\in\mathcal{J}(M,k+2)} v_{1,i}^s\, \partial_\theta v_j^s + \sum_{k=N}^{2(N+1)+(M-1)} \varepsilon^k \sum_{(i,j)\in\mathcal{J}(M,k+1)} v_{2,i}^s \partial_y v_j^s \\
&- \begin{pmatrix} \mu \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta\theta} v_{1,k}^s + \lambda\, \varepsilon^N \partial_{\theta\theta} v_{1,N+1}^s + \lambda \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta y} v_{2,k}^s \\ \mu \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta\theta} v_{2,k}^s + \lambda \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta y} v_{1,k}^s + \lambda \sum_{k=N}^{N+2} \varepsilon^k \partial_{yy} v_{2,k-1}^s. \end{pmatrix}.
\end{aligned}
\tag{90}
$$

First we can factorize by ε^N in the above expression. Then, since v_k^s is in $\mathcal{H}_T^\infty$ for $k \in [\![0, N+1]\!]$, we estimate (90) in H^m-norm and get for all integer $m \geq 2$:

$$\sup_{\varepsilon\in]0,1]} \varepsilon^{-N} \sup_{t\in[0,T]} \|R_\varepsilon^s(t,\cdot)\|_{H^m(\mathbb{T}\times\mathbb{R})} < +\infty.$$

Thus v_ε^s is an approximated solution for the operators $\mathcal{L}^{as}$ (up to order N).

∘ We constructed v_ε^f so that it approximates the operator $\mathcal{L}^{aft}$ instead of $\mathcal{L}^{af}$. To pass from $\mathcal{L}^{af}$ to $\mathcal{L}^{aft}$, we perform a Taylor formula (with respect to t up to order $N-1$):

$$\mathcal{L}^{af}(\varepsilon, v_\varepsilon^f, v_\varepsilon^s) = \mathcal{L}^{aft}(\varepsilon, v_\varepsilon^f) + R_\varepsilon^{tay},$$

where the remainder is defined as:

$$
\begin{aligned}
R_\varepsilon^{tay}(t,.) := \varepsilon^{M-2} \Bigg(&\int_0^t \frac{u^N}{N!} (\partial_t)^{N-1} v_\varepsilon^{1s}(u,.)\, du\ \partial_\theta v_\varepsilon^f(t/\varepsilon^2,\cdot) \\
&+ \varepsilon \int_0^t \frac{u^N}{N!} (\partial_t)^{N-1} v_\varepsilon^{2s}(u,.)\, du\ \partial_y v_\varepsilon^f(t/\varepsilon^2,\cdot) \Bigg) \\
+ \varepsilon^{M-2} \Bigg(&v_\varepsilon^f(t/\varepsilon^2,\cdot) \int_0^t \frac{u^N}{N!} (\partial_t)^{N-1} \partial_\theta v_\varepsilon^s(u,\cdot) du \\
&+ v_\varepsilon^f(t/\varepsilon^2,\cdot) \int_0^t \frac{u^N}{N!} (\partial_t)^{N-1} \partial_y v_\varepsilon^s(u,\cdot) du \Bigg).
\end{aligned}
$$

According to Proposition 1.1, the hypothesis of Lemma 2.10 are satisfied. For all integer $m \geq 2$ we have:

$$\sup_{\varepsilon\in]0,1]} \sup_{t\in[0,T]} \varepsilon^{-N} \left\|R_\varepsilon^{tay}(t,\cdot)\right\|_{H^m(\mathbb{T}\times\mathbb{R})} < +\infty.$$

∘ Finally, we plug $v_\varepsilon^s = \sum_{k=0}^{N+1} \varepsilon^k v_k^s$ and $v_\varepsilon^f = \sum_{k=0}^{N+1} \varepsilon^k v_k^f$ into $\mathcal{L}^{aft}$ and use the cascade of equations (43). We obtain:

$$\mathcal{L}^{aft}(\varepsilon, v_\varepsilon^f)(t/\varepsilon^2) = R_\varepsilon^{1f} + R_\varepsilon^{2f}.$$

The remainder R_ε^{1f} is defined as

$$R_\varepsilon^{1f} := \varepsilon^N \partial_y v_{N+1}^f(t/\varepsilon^2,\cdot) + \sum_{k=N}^{2(N+1)+(M-2)} \varepsilon^k \sum_{(i,j)\in\mathcal{J}(M,k+2)} v_{1,i}^f(t/\varepsilon^2,\cdot)\,\partial_\theta v_j^f(t/\varepsilon^2,\cdot)$$
$$+ \sum_{k=N}^{2(N+1)+(M-1)} \varepsilon^k \sum_{(i,j)\in\mathcal{J}(M+1,k+2)} v_{2,i}^f(t/\varepsilon^2,\cdot)\partial_y v_j^f(t/\varepsilon^2,\cdot)$$
$$- \begin{pmatrix} \mu \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta\theta} v_{1,k}^f(t/\varepsilon^2,\cdot) + \lambda\,\varepsilon^N \partial_{\theta\theta} v_{1,N+1}^f(t/\varepsilon^2,\cdot) + \lambda \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta y} v_{2,k}^f(t/\varepsilon^2,\cdot) \\ \mu \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta\theta} v_{2,k}^f(t/\varepsilon^2,\cdot) + \lambda \sum_{k=N}^{N+1} \varepsilon^k \partial_{\theta y} v_{1,k}^f(t/\varepsilon^2,\cdot) + \lambda \sum_{k=N}^{N+2} \varepsilon^k \partial_{yy} v_{2,k-1}^f(t/\varepsilon^2,\cdot) \end{pmatrix},$$

whereas R_ε^{2f} consists of the terms corresponding to the Taylor formula:

$$R_\varepsilon^{2f} := \sum_{k=N}^{4N+(M-2)} \varepsilon^k \Bigg(\sum_{(i,j,l)\in\mathcal{I}(M,k+2)} \frac{t^l}{\varepsilon^{2l} l!} \partial_t^l v_{1,i}^s(0,\cdot)\,\partial_\theta v_j^f(t/\varepsilon^2,\cdot)$$
$$+ \sum_{(i,j,l)\in\mathcal{I}(M,k+2)} \frac{t^l}{\varepsilon^{2l} l!} v_{1,i}^f(t/\varepsilon^2,\cdot)\partial_t^l \partial_\theta v_j^s(0,\cdot) \Bigg)$$
$$+ \sum_{k=N}^{4N+(M-1)} \varepsilon^k \Bigg(\sum_{(i,j,l)\in\mathcal{I}(M,k+1)} \frac{t^l}{\varepsilon^{2l} l!} \partial_t^l v_{2,i}^s(0,\cdot)\,\partial_y v_j^f(t/\varepsilon^2,\cdot)$$
$$+ \sum_{(i,j,l)\in\mathcal{I}(M,k+1)} \frac{t^l}{\varepsilon^{2l} l!} v_{1,i}^f(t/\varepsilon^2,\cdot)\partial_t^l \partial_y v_j^s(0,\cdot) \Bigg).$$

-By construction (see Proposition 1.1), the profiles $\{v_k^f\}_{k\in[\![0,N+1]\!]}$ lie in $\mathcal{E}_\delta^\infty$. Furthermore, we can factorize R_ε^{1f} by ε^N. We deduce that for all integer $m \geq 2$:

$$\sup_{\varepsilon\in]0,1]} \sup_{t\in[0,T]} \varepsilon^{-N} \left\| R_\varepsilon^{1f}(t,\cdot) \right\|_{H^m(\mathbb{T}\times\mathbb{R})} < +\infty.$$

-There remains to estimate the term R_ε^{2f}. *A priori* this term can be dangerous because it contains some polynomials in the variable t/ε^2. Nevertheless we can use the fast decreasing behaviours of the profile v_ε^f. Indeed if $f \in \mathcal{E}_\delta^\infty$ then for all $l \in \mathbb{N}$ the function $\tau^l f$ is also quickly decreasing: $\tau^l f \in \mathcal{E}_{\delta'}^\infty$ for some $0 < \delta' < \delta$. Furthermore, noticing than we can factorize by ε^N in R_ε^{2f}, we obtain for all integer $m \geq 2$:

$$\sup_{\varepsilon\in]0,1]} \sup_{t\in[0,T]} \varepsilon^{-N} \left\| R_\varepsilon^{2f}(t,\cdot) \right\|_{H^m(\mathbb{T}\times\mathbb{R})} < +\infty.$$

□

2.2 The case of the pressure - Consequence

Here, we still assume that $M \geq 2$. First of all we quickly prove Proposition . Then we take advantage of the control obtained on $\{q_\varepsilon^a\}_\varepsilon$ to prove that the approximated solution v_ε^a is a good approximation for operator $(\mathcal{L}^1, \mathcal{L}^2)$ assuming ν is large enough (Proposition 1.3).

Approximated pressure. Consider the approximated velocity v_ε^a built according to Proposition 1.1. Since the operator $\mathcal{L}_0$ is linear with respect to the pressure variable, we build the profile q_k^ε as the solution of the following problem:

$$\mathcal{L}_0(\varepsilon, q_k^\varepsilon, v_\varepsilon^a) = \partial_t q_k^\varepsilon + \varepsilon^{-1} h\,\partial_y q_k^\varepsilon \\ + \varepsilon^{M-2}(v_\varepsilon^{a1}\partial_\theta q_k^\varepsilon + \varepsilon\, v_\varepsilon^{2a}\partial_y q_k^\varepsilon) + C\,\varepsilon^{M-2} q_k^\varepsilon(\partial_\theta v_\varepsilon^{a1} + \varepsilon \partial_y v_\varepsilon^{2a}) = 0 \tag{91}$$

with initial data satisfying $q_k^\varepsilon(0,\cdot) = q_k^0(\cdot)$.

At fixed ε, for any positive time T, (91) has an unique solution q_k^ε in $\mathcal{H}_T^m(\mathbb{T}\times\mathbb{R})$. We recover the approximated solution q_ε^a (on the strip $[0,T]$) by summing over all the multi-indices $k \in [\![0, N+1]\!]$:

$$q_\varepsilon^a := \sum_{k=0}^{N+1} \varepsilon^k q_k^\varepsilon.$$

However, since the transport is singular the family of solution $\{q_\varepsilon^a\}_\varepsilon$ is not bounded in $\mathcal{H}_T^m(\mathbb{T}\times\mathbb{R})$. Yet, it is in the anisotropic Sobolev spaces. In other words, Inequality (14) is satisfied.

Approximated solution for the operator $\mathcal{L}$. A direct consequence of Inequality (14) is that $\{(q_\varepsilon^a, v_\varepsilon^a)\}_\varepsilon$ is an approximated solution for the operator $\mathcal{L}$, *i.e.* Proposition 1.3 is satisfied.

First, we have $\mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a) = 0$ and:

$${}^t\left(\mathcal{L}_1, \mathcal{L}_2\right)(\varepsilon, q_\varepsilon^a, v_\varepsilon^a) = \mathcal{L}^a(\varepsilon, v_\varepsilon^a) + C\varepsilon^{2\nu-M-2}\ {}^t\left(q_\varepsilon^a\partial_\theta q_\varepsilon^a, \varepsilon\, q_\varepsilon^a\partial_y q_\varepsilon^a\right).$$

The quantity $\mathcal{L}^a(\varepsilon, v_\varepsilon^a)$ can be estimated thanks to (11). The pressure term $\varepsilon^{2\mathcal{V}-M-2}\,{}^t(q_\varepsilon^a\partial_\theta q_\varepsilon^a, \varepsilon\, q_\varepsilon^a \partial_y q_\varepsilon^a)$ can be estimated thanks to the Gagliardo-Niremberg's Inequality. Let $\alpha \in \mathbb{N}^2$, $|\alpha| \le m$, then

$$\left\|\partial^\alpha\left(q_\varepsilon^a\partial_\theta q_\varepsilon^a\right)\right\|_{L^2} = \left\|\partial^\alpha\partial_\theta\left(q_\varepsilon^a\right)^2\right\|_{L^2} \le 2C_g \left\|q_\varepsilon^a\right\|_{L^\infty}\left\|q_\varepsilon^a\right\|_{\mathring{H}^{|\alpha|+1}}.$$

To recover an estimate in the anisotropic version of the Sobolev spaces, we use (5) together with the following equivalence of norms:

$$\|\cdot\|_{H^m} \le \varepsilon^{-m}\|\cdot\|_{H^m_{(1,\varepsilon)}}. \tag{92}$$

Thus, we obtain

$$\varepsilon^{-N}\left\|\varepsilon^{2\nu-M-2}\,{}^t(q_\varepsilon^a\partial_\theta q_\varepsilon^a, \varepsilon\, q_\varepsilon^a\partial_y q_\varepsilon^a)\right\|_{H^m} \lesssim \varepsilon^{2\nu-M-5/2-(m+1)-N}\left\|q_\varepsilon^a\right\|^2_{H^{m+1}_{(1,\varepsilon)}} \lesssim \varepsilon^{2\nu-M-5/2-(m+1)-N}.$$

Assuming (15), it completes the proof. ∎

3 Energy estimates

In this section we prove Theorem 1.4. The integers m, ν, M, N and R satisfy property (18). As ever mentioned at the level of the introduction, page 12, the main issue is to get a control over the singular term $\varepsilon^{-2}\partial_\theta h\, v_\varepsilon^{1R}$. To desingularize it, we consider the new unknowns:

$$\widetilde{q}_\varepsilon^R := q_\varepsilon^R, \qquad \widetilde{v}_\varepsilon^{1R} := v_\varepsilon^{1R}, \qquad \widetilde{v}_\varepsilon^{2R} := \varepsilon v_\varepsilon^{2R}. \tag{93}$$

It satisfies a hyperbolic-parabolic system (singular in ε):

$$\begin{aligned}
&\partial_t \widetilde{q}_\varepsilon^R + \varepsilon^{-1} h\, \partial_y \widetilde{q}_\varepsilon^R + \varepsilon^{M-2}\left(v_\varepsilon^{1a}\, \partial_\theta \widetilde{q}_\varepsilon^R + \varepsilon v_\varepsilon^{2a}\, \partial_y \widetilde{q}_\varepsilon^R\right) + C\, \varepsilon^{M-2} \widetilde{q}_\varepsilon^R \left(\partial_\theta v_\varepsilon^{1a} + \varepsilon\, \partial_y v_\varepsilon^{2a}\right) \\
&\qquad + \varepsilon^{M-2}\left(\widetilde{v}_\varepsilon^{1R}\, \partial_\theta q_\varepsilon^a + \widetilde{v}_\varepsilon^{2R}\, \partial_y q_\varepsilon^a\right) + C\, \varepsilon^{M-2} q_\varepsilon^a \left(\partial_\theta \widetilde{v}_\varepsilon^{1R} + \partial_y \widetilde{v}_\varepsilon^{2R}\right) = \widetilde{S}_\varepsilon^{0,R,N}, \qquad (94a) \\
&\partial_t \widetilde{v}_\varepsilon^R + \varepsilon^{-1} h\, \partial_y \widetilde{v}_\varepsilon^R + \boxed{{}^t\left(0, \varepsilon^{-1}\, \partial_\theta h\, \widetilde{v}_\varepsilon^{1R}\right)} + \varepsilon^{M-2}\left(v_\varepsilon^{1a}\, \partial_\theta \widetilde{v}_\varepsilon^R + \varepsilon v_\varepsilon^{2a}\, \partial_y \widetilde{v}_\varepsilon^R\right) \\
&\qquad + \varepsilon^{M-2}\left(\widetilde{v}_\varepsilon^{1R} \partial_\theta + \widetilde{v}_\varepsilon^{2R} \partial_y\right) {}^t(v_\varepsilon^{1a}, \varepsilon\, v_\varepsilon^{2a}) - \mathcal{Q}_\varepsilon(\widetilde{v}_\varepsilon^R) = \widetilde{S}_\varepsilon^{R,N}, \qquad (94b)
\end{aligned}$$

together with the initial data $\left(q_\varepsilon^R(0,\cdot), v_\varepsilon^R(0,\cdot)\right) \equiv 0$. The right-hand side of Equation (94) is defined as:

$$\begin{aligned}
\widetilde{S}_\varepsilon^{0,R,N} &:= -\varepsilon^{N-R}(\varepsilon^{-N} \mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a)) - \varepsilon^{R+M-2}\left(\widetilde{v}_\varepsilon^{1R} \partial_\theta \widetilde{q}_\varepsilon^R + \widetilde{v}_\varepsilon^{2R} \partial_y \widetilde{q}_\varepsilon^R\right) \\
&\qquad - C \varepsilon^{R+M-2} \widetilde{q}_\varepsilon^R \left(\partial_\theta \widetilde{v}_\varepsilon^{1R} + \partial_y \widetilde{v}_\varepsilon^{2R}\right), \quad (95) \\
\widetilde{S}_\varepsilon^{R,N} &:= -\varepsilon^{N-R}\left(\varepsilon^{-N}\, {}^t\left(\mathcal{L}_1^a(\varepsilon, v_\varepsilon), \varepsilon \mathcal{L}_2^a(\varepsilon, v_\varepsilon)\right)\right) - \varepsilon^{R+M-2}\left(\widetilde{v}_\varepsilon^{1R} \partial_\theta \widetilde{v}_\varepsilon^R + \widetilde{v}_\varepsilon^{2R} \partial_y \widetilde{v}_\varepsilon^R\right) \\
&\qquad - \frac{C \varepsilon^{2\nu - M - R - 2}}{2}\, {}^t\left(\partial_\theta, \varepsilon^2 \partial_y\right) \left(q_\varepsilon^a + \varepsilon^R \widetilde{q}_\varepsilon^R\right)^2. \quad (96)
\end{aligned}$$

This is clearly a non-linear problem and the variable of pressure $\widetilde{q}_\varepsilon^R$ and the variables of velocity $\widetilde{v}_\varepsilon^R$ are coupled. The dissipation is turned into

$$\mathcal{Q}_\varepsilon \widetilde{v}_\varepsilon^R := \begin{pmatrix} \mathcal{Q}_\varepsilon^1 \widetilde{v}_\varepsilon^R \\ \mathcal{Q}_\varepsilon^2 \widetilde{v}_\varepsilon^R \end{pmatrix} = \frac{1}{\varepsilon^2} \begin{pmatrix} \lambda\left(\partial_{\theta\theta} \widetilde{v}_\varepsilon^{1R} + \varepsilon^2 \partial_{yy} \widetilde{v}_\varepsilon^{1R}\right) + \mu\varepsilon\left(\partial_{\theta\theta} \widetilde{v}_\varepsilon^{1R} + \partial_{y\theta} \widetilde{v}_\varepsilon^{2R}\right) \\ \lambda\left(\partial_{\theta\theta} \widetilde{v}_\varepsilon^{2R} + \varepsilon^2 \partial_{yy} \widetilde{v}_\varepsilon^{2R}\right) + \varepsilon^2 \mu\varepsilon\left(\partial_{\theta y} \widetilde{v}_\varepsilon^{1R} + \partial_{yy} \widetilde{v}_\varepsilon^{2R}\right) \end{pmatrix}. \qquad (97)$$

We clearly desingularize the hyperbolic part to a cost on the parabolic part, $\mathcal{Q}_\varepsilon$. It can no longer satisfy an estimate such as (28) necessary to keep control over singular terms in (94). One important aspect of the proof is to obtain Inequality (108). If it is not satisfied, the mechanism of the proof fails.

Thus, in this section, we look for accurate energy estimates for the unknown $(\widetilde{q}_\varepsilon^R, \widetilde{v}_\varepsilon^R)$. We easily go back to the initial variables according to (93).

Coupling and nonlinear aspects. In Equation (94b), the variables of pressure and velocity are only coupled threw the term:

$$\frac{C \varepsilon^{2\nu - M - R - 2}}{2}\, {}^t(\partial_\theta, \varepsilon^2\, \partial_y) \left(q_\varepsilon^a + \varepsilon^R \widetilde{q}_\varepsilon^R\right)^2. \qquad (98)$$

If ν is large enough, the Equation (94b) is somehow independent of the pressure. It seems that we can deal with the velocity and then deal with the pressure. This is reinforced by the fact that the pressure and the velocity are estimated in different Sobolev Spaces.

Yet, the term (98) has to be estimated carefully. We have to link the anisotropic Sobolev norms with the classical Sobolev norms (92). It explains the loss of precision on w_m with respect to the regularity m (see Equation (18)).

In practice, we perform energy estimates on Equation (94b). Then we plug the estimates obtained on the velocity into Equation (94a). We underline here the fact that we need more regularity on the velocity to estimate the term $\widetilde{S}_\varepsilon^{0,R,N}$ (defined by Equation (95)). Thus, the *regularization* of the velocity (thanks to the dissipation) plays again a crucial role.

Finally, the problem is nonlinear (for example term (98)). Classically, nonlinear terms are estimated thanks to the Gagliardo-Niremberg's estimate which required a L^∞ control over the unknowns. To get this control, we introduce the characterized time T_ε^*:

$$T_\varepsilon^* := \min\left(1, \sup_{T\in[0,T_\varepsilon]} \left\{\forall\, t \in [0,T],\ \|\widetilde{q}_\varepsilon^R(t,\cdot)\|_{H^{m+3}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} \leq 2,\ \|\widetilde{v}_\varepsilon^R(t,\cdot)\|_{H^{m+3}(\mathbb{T}\times\mathbb{R})} \leq 2\right\}\right).$$

It provides L^∞-estimates on the strip $[0,T_\varepsilon^*]$, for all $\varepsilon \in]0,1]$ (see (5)):

$$\forall\, t\in[0,T_\varepsilon^*], \quad \|\sqrt{\varepsilon}\,\widetilde{q}_\varepsilon^R(t,\cdot)\|_{W^{m+1,\infty}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} \leq 2, \qquad \|\widetilde{v}_\varepsilon^R(t,\cdot)\|_{W^{m+1,\infty}(\mathbb{T}\times\mathbb{R})} \leq 2. \tag{99}$$

3.0.1 Results and Consequences

As indicated, we first prove an estimate over the velocity on the interval $[0,T_\varepsilon^*]$, thanks to Equation (94b):

Proposition 3.1. *Let $\widetilde{v}_\varepsilon^R$ be a solution of (94) on $[0,T_\varepsilon^*]$. There exist a positive constant ε_{crit} and two positive constants K_m^1 and K_m^2 (independent of ε) such that*

$$\forall\,\varepsilon\in]0,\varepsilon_{crit}], \quad \forall\, t\in[0,T_\varepsilon^*], \qquad \|\widetilde{v}_\varepsilon^R(t,\cdot)\|^2_{H^{m+3}(\mathbb{T}\times\mathbb{R})} \leq \varepsilon^{2w_m}\, K_m^1\left(e^{K_m^2\, t} - 1\right). \tag{100}$$

Furthermore, one can prove a regularization property. Select a multi-index $\alpha\in\mathbb{N}^2$ of length $|\alpha|$ smaller than $m+4$; then

$$\forall\,\varepsilon\in]0,\varepsilon_{crit}],\ \forall\, t\in[0,T_\varepsilon^*], \quad \int_0^t \left\|\varepsilon^{-(1-\delta_{\alpha_1,0})}\partial^\alpha \widetilde{v}_\varepsilon^R(s,\cdot)\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})} ds \leq K_m^1\,\varepsilon^{2w_m} t\,, \tag{101}$$

where $\delta_{i,j}$ denotes the Kronecker symbol (of two integers): $\delta_{i,j}=0$ if $i\neq j$ and $\delta_{i,i}=1$.

By plugging (100)-(101) in (94a) we obtain:

Proposition 3.2. *Let $\widetilde{q}_\varepsilon^R$ be a solution of Equation (94) on $[0,T_\varepsilon^*]$. There exist a positive constant ε_{crit} and a positive constant K_m^1 (independent of ε) such that*

$$\forall\,\varepsilon\in]0,\varepsilon_{crit}], \quad \forall\, t\in[0,T_\varepsilon^*], \qquad \|\widetilde{q}_\varepsilon^R(t,\cdot)\|^2_{H^{m+3}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} \leq \varepsilon^{2w_m} K_m^1\, t.$$

Those two lemmas allow to prove an accurate estimate of the solution on a strip independent of ε.

Corollary 3.3. *Consider integers m, ν, M, N and R satisfying the condition* (18)*. Then, there exist two positive constants ε_c and T_c (independent of ε), and a positive constant $c_{err} > 0$ such that,*

$$\forall\, t\in[0,T_c], \quad \forall\,\varepsilon\in]0,\varepsilon_c], \qquad \|\widetilde{q}_\varepsilon^R(t,\cdot)\|_{H^{m+3}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} \leq c_{err}, \qquad \|\widetilde{v}_\varepsilon^R(t,\cdot)\|_{H^{m+3}(\mathbb{T}\times\mathbb{R})} \leq c_{err}.$$

Proof of Corollary 3.3. We argue by contradiction:

$$\forall\,(\widetilde{\varepsilon},T)\in]0,1]\times[0,1], \quad \exists\,\varepsilon\in]0,\widetilde{\varepsilon}],\ T_\varepsilon^* < T.$$

We recall that there exist ε_d and C_m positive constants such that, for all $\varepsilon\in]0,\varepsilon_d]$ and for all time $t\in[0,T_\varepsilon^*]$:

$$\|\widetilde{q}_\varepsilon^R(t,\cdot)\|^2_{H^{m+3}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} \leq \varepsilon^{2w_m}\, C_m\, t, \qquad \|\widetilde{v}_\varepsilon^R(t,\cdot)\|^2_{H^{m+3}(\mathbb{T}\times\mathbb{R})} \leq \varepsilon^{2w_m} C_m\, t\,.$$

We choose for instance $T = \min(\frac{1}{2\,C_m}, \frac{1}{2})$ and $\widetilde{\varepsilon} = \varepsilon_d < 1$. In particular, $T < 1$. From assumption, there exists $\varepsilon_0 \in]0, \varepsilon_d]$ such that $T^*_{\varepsilon_0} < T$. Furthermore, since $w_m \geq 0$ from condition (18) we get:

$$\forall\, t \in [0, T^*_{\varepsilon_0}], \qquad \left\|\widetilde{q}^R_{\varepsilon_0}(t,\cdot)\right\|_{H^{m+3}_{(1,\varepsilon_0)}(\mathbb{T}\times\mathbb{R})} \leq \frac{1}{2} < 2, \qquad \left\|\widetilde{v}^R_{\varepsilon_0}(t,\cdot)\right\|_{H^{s+3}(\mathbb{T}\times\mathbb{R})} \leq \frac{1}{2} < 2.$$

Now consider the applications

$$t \in [0, T^*_{\varepsilon_0}[\longmapsto \left\|\widetilde{q}^R_{\varepsilon_0}(t,\cdot)\right\|_{H^{m+3}_{(1,\varepsilon_0)}(\mathbb{T}\times\mathbb{R})} \quad \text{and} \quad t \in [0, T^*_{\varepsilon_0}[\longmapsto \left\|\widetilde{v}^R_{\varepsilon_0}(t,\cdot)\right\|_{H^{m+3}(\mathbb{T}\times\mathbb{R})}.$$

They are continuous. $\widetilde{q}^R_{\varepsilon_0}$ (respectively $\widetilde{v}^R_{\varepsilon_0}$) can be extended in time as long as the quantity $\left\|\widetilde{q}^R_{\varepsilon_0}(t,\cdot)\right\|_{H^{m+3}_{(1,\varepsilon_0)}}$ (respectively $\left\|\widetilde{v}^R_{\varepsilon_0}(t,\cdot)\right\|_{H^{m+3}}$) remains bounded.

It follows that we can find $T \in]T^*_{\varepsilon_0}, T_{\varepsilon_0}[$ such that for all time $t \in [0, T]$, $\left\|\widetilde{q}^R_{\varepsilon_0}(t,.)\right\|_{H^{m+3}_{(1,\varepsilon_0)}} < 2$ (respectively $\left\|\widetilde{v}^R_{\varepsilon_0}(t,.)\right\|_{H^{m+3}} < 2$). This is in contradiction with the definition of $T^*_{\varepsilon_0}$. $\square$

Assuming Propositions 3.1 and 3.2 are satisfied, Corollary 3.3 holds. We go back to the initial unknowns thanks to the Equation (93) and obtain Theorem 1.4.

Thus in Subsection 3.1, we prove Proposition 3.1. We take advantage of Inequalities (100) and (101) to obtain Proposition 3.2.

3.0.2 Notations

Here we introduce some notations required for the proof of Propositions 3.1 and 3.2.

Let $(q^a_\varepsilon, v^a_\varepsilon)$ an approximated solution of order N constructed on the interval $[0,1]$ according to Proposition 1.1 and Proposition 1.2. The family $\{(q^a_\varepsilon, v^a_\varepsilon)\}_\varepsilon$ lies in $\mathcal{H}^{m+6,0}_{1,(1,\varepsilon)} \times \mathcal{H}^{m+6,0}_1$ and satisfies Inequalities (11) and (14). We denote by C_a, $C_{\mathcal{L}^a}$ and $C_{\mathcal{L}}$ positive constants such that:

$$\sup_{\varepsilon\in]0,1]} \sup_{t\in[0,1]} \|v^a_\varepsilon(t,\cdot)\|_{H^{m+6}} < C_a, \qquad \sup_{\varepsilon\in]0,1]} \sup_{t\in[0,1]} \|q^a_\varepsilon(t,\cdot)\|_{H^{m+6}_{(1,\varepsilon)}} < C_a, \tag{102}$$

and

$$\sup_{\varepsilon\in\,]0,1]} \sup_{t\in[0,1]} \left\|\varepsilon^{-N}\,\mathcal{L}^a(\varepsilon, v^a_\varepsilon)\right\|_{H^{m+6}(\mathbb{T}\times\mathbb{R})} < C_{\mathcal{L}^a}\,, \tag{103}$$

$$\sup_{\varepsilon\in\,]0,1]} \sup_{t\in[0,1]} \left\|\varepsilon^{-N}\,\mathcal{L}_0(\varepsilon, q^a_\varepsilon, v^a_\varepsilon)\right\|_{H^{m+6}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} < C_{\mathcal{L}}\,. \tag{104}$$

In what follows we adopt the conventions:

$$\forall\, m \in \mathbb{N}^*, \qquad \|f\|_{H^{-m}} \equiv 0,$$

to simplify all the statement.

3.1 Energy estimates for the velocity

In this section, we prove Proposition 3.1 by induction on the size of m settings:

$$\mathcal{P}(m)\ :\ \text{" Proposition 3.1 holds up to the integer } m\text{".} \tag{105}$$

To go from m to $m+1$, we prove an *energy inequality* for the velocity performing an energy method on Equation (94b) in the homogeneous Sobolev space $\overset{\circ}{H}{}^m$ (defined page 3).

Lemma 3.4. *There exist c_1 and ε_d two positive constants (which only depend on m) such that for any $J \in [\![0, m+3]\!]$, there exist four positive constants C_p, C_J^1, C_J^2 and C_J^3 such that for all $\varepsilon \in]0, \varepsilon_d]$, and for all time $t \in [0, T_\varepsilon^*]$,*

$$\frac{1}{2}\partial_t \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\mathring{H}^J(\mathbb{T}\times\mathbb{R})} + c_1 \sum_{|\alpha|=J} \Phi_\varepsilon(\nabla, \partial^\alpha \widetilde{v}_\varepsilon^R)(t) \leq C_J^1 \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\mathring{H}^J(\mathbb{T}\times\mathbb{R})} + (J+1)C_p\varepsilon^{2w_m}$$
$$+ C_J^2 \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^{J-1}(\mathbb{T}\times\mathbb{R})} + C_J^3 \sum_{|\alpha|=J} \left\|\varepsilon^{-(1-\delta_{\alpha_1,0})}\partial^\alpha \widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})}.$$

Subsections 3.1.1-...-3.1.3 are dedicated to the proof of Lemma 3.4. Then in Subsection 3.1.4, we finally prove Proposition 3.1.

First, consider a multi-index $\alpha \in \mathbb{N}^2$ and differentiate α times Equation (). Then we multiply by $\partial^\alpha \widetilde{v}_\varepsilon^R$ and integrate (with respect to the space variables θ and y),

$$\frac{1}{2}\partial_t \left\|\partial^\alpha \widetilde{v}_\varepsilon^R\right\|^2_{L^2} - \left\langle \partial^\alpha(\mathcal{Q}_\varepsilon \widetilde{v}_\varepsilon^R), \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle = \left\langle \partial^\alpha S_\varepsilon^{R,N}, \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle - \left\langle \partial^\alpha(\mathcal{A}\widetilde{v}_\varepsilon^R), \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle$$
$$- \left\langle \partial^\alpha(\mathcal{B}\widetilde{v}_\varepsilon^R), \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle - \left\langle \partial^\alpha(\mathcal{C}\widetilde{v}_\varepsilon^R), \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle - \left\langle \partial^\alpha(\mathcal{H}\widetilde{v}_\varepsilon^R), \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle, \quad (106)$$

where the operators $\mathcal{H}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are defined as follows:

$$\mathcal{H}\,\widetilde{v}_\varepsilon^R := \varepsilon^{-1} h\, \partial_y \widetilde{v}_\varepsilon^R, \qquad \mathcal{A}\,\widetilde{v}_\varepsilon^R := \varepsilon^{M-2}\left(\widetilde{v}_\varepsilon^{1R}\partial_\theta + \widetilde{v}_\varepsilon^{2R}\partial_y\right)\,{}^t(v_\varepsilon^{1a}, \varepsilon\, v_\varepsilon^{2a}),$$
$$\mathcal{C}\,\widetilde{v}_\varepsilon^R :=^t \left(0, \varepsilon^{-1}\partial_\theta h\, \widetilde{v}_\varepsilon^{1R}\right), \quad \mathcal{B}\,\widetilde{v}_\varepsilon^R := \varepsilon^{M-2}\left(v_\varepsilon^{1a}\partial_\theta \widetilde{v}_\varepsilon^R + \varepsilon v_\varepsilon^{2a}\partial_y \widetilde{v}_\varepsilon^R\right).$$

The strategy of the estimates is the following. We first prove the coercive estimate over $\mathcal{Q}_\varepsilon$ (see Lemma 3.5 in Subsection 3.1.1). Then we compute all the singular contribution (with respect to the regularity and to ε) in Lemmas 3.6-...-3.8 that we absorb thanks to $\mathcal{Q}_\varepsilon$ (see Subsection 3.1.2). Finally, we simply bound the remaining terms thanks to Lemma 3.9 in Subsection 3.1.3.

3.1.1 Step one : Coercive estimate for the dissipation

We start by estimating the term involving the dissipation $\mathcal{Q}_\varepsilon$.

Lemma 3.5. *We recall that $\mu > 0$. Select constant λ and μ which satisfy*

$$\lambda < 4\,\mu\,. \quad (107)$$

There exists $c_0 > 0$, there exists $\varepsilon_d \in]0, 1]$, such that for any function $f \in H^1(\mathbb{T}\times\mathbb{R}, \mathbb{R}^2)$:

$$\forall\, \varepsilon \in]0, \varepsilon_d], \qquad -\langle \mathcal{Q}_\varepsilon f, f\rangle \geq c_0\ \Phi_\varepsilon(\nabla, f)\,. \quad (108)$$

where Φ_ε is defined in Equation (28).

Proof of Lemma 3.5. Let $f \in H^1(\mathbb{T}\times\mathbb{R}, \mathbb{R}^2)$. We decompose f in Fourier series (in θ) into:

$$f(\theta, y) := \sum_{k\in\mathbb{Z}} f_k(y) e^{ik\theta}\,,$$

then we use a Fourier transform in the variable y. By the Parseval equality, up to a constant (that we forget here),

$$-\langle \mathcal{Q}_\varepsilon f, f\rangle = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \widehat{f}_k(\xi) Q_\varepsilon(k,\xi) \widehat{f}_k(\xi) d\xi\,, \qquad \widehat{f}_k(\xi) := \int_{\mathbb{R}} e^{i\,y\cdot\xi} f_k(y)\, dy\,,$$

where Q_ε is defined as:

$$Q_\varepsilon(k,\xi) := \begin{pmatrix} (\mu+\lambda\varepsilon)\frac{k^2}{\varepsilon^2}+\xi^2 & \frac{\lambda\varepsilon}{2}\left(1+\frac{1}{\varepsilon^2}\right)k\,\xi \\ \frac{\lambda\varepsilon}{2}\left(1+\frac{1}{\varepsilon^2}\right)k\,\xi & \mu\frac{k^2}{\varepsilon^2}+(\mu+\lambda\varepsilon)\xi^2 \end{pmatrix}, \quad k\in\mathbb{Z}, \quad \xi\in\mathbb{R}.$$

We interpret $\langle \mathcal{Q}_\varepsilon f, f\rangle$ as a quadratic form in the variables $\widehat{f}_k^1$ and $\widehat{f}_k^2$. In this way, to prove Inequality (108) we show that there exist ε_d and c_0 two positive constants such that for all $g\in H^1(\mathbb{R},\mathbb{R}^2)$:

$$\forall\, k\in\mathbb{Z}, \quad \forall\,\xi\in\mathbb{R}, \qquad \widehat{g}(\xi)Q_\varepsilon(k,\xi)\widehat{g}(\xi) \geq c_0\left(\varepsilon^{-2}\,k^2+\xi^2\right)\,\left(\widehat{g}_1^2(\xi)+\widehat{g}_2^2(\xi)\right). \tag{109}$$

At fixed $(k,\xi)\in\mathbb{Z}\times\mathbb{R}$, $Q_\varepsilon(k,\xi)$ is a diagonalizable matrix since real and symmetric. We compute its eigenvalues:

$$\mu_\varepsilon^1(k,\xi) := \frac{2\mu+\lambda\varepsilon}{2}((\varepsilon^{-1}k)^2+\xi^2)+\frac{\lambda\varepsilon}{2}\sqrt{((\varepsilon^{-1}k)^4+\xi^4)+(\varepsilon(\varepsilon^{-1}k)\xi)^2+\varepsilon^{-2}\left(\varepsilon^{-1}k\xi\right)^2},$$
$$\mu_\varepsilon^2(k,\xi) := \frac{2\mu+\lambda\varepsilon}{2}((\varepsilon^{-1}k)^2+\xi^2)-\frac{\lambda\varepsilon}{2}\sqrt{((\varepsilon^{-1}k)^4+\xi^4)+(\varepsilon(\varepsilon^{-1}k)\xi)^2+\varepsilon^{-2}\left(\varepsilon^{-1}k\xi\right)^2}\,.$$

In the end of the proof, we show that for any $(k,\xi)\in\mathbb{Z}\times\mathbb{R}$ we have

$$\mu_\varepsilon^2(k,\xi)\geq c_0\left(\varepsilon^{-2}\,k^2+\xi^2\right)\,\left(\widehat{g}_1^2(\xi)+\widehat{g}_2^2(\xi)\right).$$

Since $\mu_\varepsilon^1\geq\mu_\varepsilon^2$, we clearly obtain Inequality (109).

∘ We define the function $\mu_\varepsilon:\mathbb{R}^2\longrightarrow\mathbb{R}$ by:

$$\mu_\varepsilon(x,y) := \frac{2\mu+\lambda\varepsilon}{2}(x^2+y^2)-\frac{\lambda\varepsilon}{2}\sqrt{(x^4+y^4)+(\varepsilon\,x\,y)^2+\varepsilon^{-2}\,(x\,y)^2}.$$

Then there exist ε_d and c_0 (independent of ε) such that

$$\forall(x,y)\in\mathbb{R}^2, \qquad \mu_\varepsilon(x,y)\geq c_0(x^2+y^2). \tag{110}$$

-First of all the function μ_ε is homogeneous of order 2 in the sense that it satisfies for all $\alpha\in\mathbb{R}$:

$$\forall\,(x,y)\in\mathbb{R}\times\mathbb{R}, \qquad \mu_\varepsilon(\alpha\,x,\alpha\,y)=\alpha^2\mu_\varepsilon(x,y)\,.$$

Thus, we prove (110) on the restricted set $(x,y)\in\mathbb{S}^1$ the sphere of center 0 and radius 1.

-We expand μ_ε as ε goes to 0^+:

$$\mu_\varepsilon(x,y)=(x^2+y^2)-\frac{\lambda}{2}|x||y|+O(\varepsilon),$$

where $O(\varepsilon)$ is uniform in $(x,y)\in\mathbb{S}^1$. Let us recall that we have:

$$\forall(x,y)\in\mathbb{R}^2, \qquad (x^2+y^2)-c|x||y|\geq 0 \qquad\Longleftrightarrow\qquad c<2.$$

That is the case if and only if $\lambda<4\mu$, *i.e.* assumption (107) is satisfied. We get an uniform bound of μ_ε (in ε). Finally, there exist ε_d and c_0 two positive constants such that Inequality (110) is satisfied.

∘ Plugging $x=\varepsilon^{-1}k$ and $y=\xi$ in Inequality (110), we obtain for all $\varepsilon\in]0,\varepsilon_d]$:

$$\forall\,(k,\xi)\in\mathbb{Z}\times\mathbb{R}\,, \qquad \mu_\varepsilon^2(k,\xi)\geq c_0\,((\varepsilon^{-1}k)^2+\xi^2).$$

We deduce that

$$\widehat{f}_k(\xi) Q_\varepsilon(k,\xi) \widehat{f}_k(\xi) \geq c_0\, ((\varepsilon^{-1}k)^2 + \xi^2) \|\widehat{f}_k(\xi)\|^2.$$

Finally, summing over $k \in \mathbb{Z}$ and integrating with respect to ξ, we have:

$$-\langle \mathcal{Q}_\varepsilon f, f\rangle \geq c_0 \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} ((\varepsilon^{-1}k)^2 + \xi^2) \|\widehat{f}_k(\xi)\|^2 d\xi = c_0\, \Phi_\varepsilon(\nabla, f).$$

□

Remark 1. *The singular change of unknowns desingularizes the hyperbolic part of system (94) to a cost on the parabolic part $\mathcal{P}_\varepsilon$. $\mathcal{Q}_\varepsilon$ is dissipative enough assuming the Δ-part of the dissipation is strong enough with respect to the ∇ div-part ($\lambda < 4\mu$).*

Remark 2. *The Inequality (108) can be proved for a more general family of dissipation. We can replace $\varepsilon\lambda$ in the dissipation $\mathcal{Q}_\varepsilon$ by some coefficient λ_ε going to 0 when ε approaches 0. Then replacing assumption (107) by:*

$$\limsup_{\varepsilon\to 0^+} \frac{\lambda_\varepsilon}{\varepsilon} < 4\mu.$$

Inequality (108) still holds.

From Equation (106) and the preceding lemma we deduce that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\frac{1}{2}\partial_t \left\|\partial^\alpha \widetilde{v}_\varepsilon^R\right\|_{L^2}^2 + c_0 \Phi_\varepsilon(\nabla, \widetilde{v}_\varepsilon^R) \leq \left\langle \partial^\alpha S_\varepsilon^{R,N}, \partial^\alpha \widetilde{v}_\varepsilon^R \right\rangle - \left\langle \partial^\alpha \left(\mathcal{A}\, \widetilde{v}_\varepsilon^R\right), \partial^\alpha \widetilde{v}_\varepsilon^R \right\rangle$$
$$- \left\langle \partial^\alpha \left(\mathcal{B} \widetilde{v}_\varepsilon^R\right), \partial^\alpha \widetilde{v}_\varepsilon^R \right\rangle - \left\langle \partial^\alpha \left(\mathcal{C} \widetilde{v}_\varepsilon^R\right), \partial^\alpha \widetilde{v}_\varepsilon^R \right\rangle - \left\langle \partial^\alpha \left(\mathcal{H} \widetilde{v}_\varepsilon^R\right), \partial^\alpha \widetilde{v}_\varepsilon^R \right\rangle. \quad (111)$$

3.1.2 Step two : estimates over the singular terms

We now try to absorb the singular terms in (106), thanks to Φ_ε. Singular terms are of two types:

-the operator only singular with respect to the regularity, $S_\varepsilon^{R,N}$,

-the operators at least singular with respect of ε, $\mathcal{H}$ and $\mathcal{C}$.

Contribution of $S_\varepsilon^{R,N}$. We start by estimating the term $S_\varepsilon^{R,N}$. It contains nonlinear terms together with the coupling terms (98). What can be underlined is that this term (98) is nonlinear (only) singular with respect to the number of derivatives acting on the pressure. This is a problem since the dissipation only regularizes the velocity. When possible, we pass those extra-derivatives onto the velocity (thanks to an integration by parts). One can prove:

Lemma 3.6. *[Control over $S_\varepsilon^{R,N}$] Select a multi-index $\alpha \in \mathbb{N}^2$ with length smaller than $m+3$ and a positive constant C_S. There exist C_S^1, C_p two positive constants such that for any $\varepsilon \in]0,1]$, for any time $t \in [0, T_\varepsilon^*]$,*

$$\left|\left\langle \partial^\alpha S_\varepsilon^{R,N}, \partial^\alpha v_\varepsilon^R \right\rangle\right|(t,\cdot) \leq C_S \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|_{\overset{\circ}{H}^{|\alpha|+1}(\mathbb{T}\times\mathbb{R})}^2 + C_S^1 \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|_{\overset{\circ}{H}^{|\alpha|}(\mathbb{T}\times\mathbb{R})}^2 + C_p\, \varepsilon^{2w_m}.$$

Proof of Lemma 3.6. Consider $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \le m+3$. We decompose the source into

$$\begin{aligned}\left\langle \partial^\alpha S_\varepsilon^{R,N}, \partial^\alpha v_\varepsilon^R \right\rangle = &-\varepsilon^{N-R} \left\langle \partial^\alpha \left(\varepsilon^{-N} \mathcal{L}^a(\varepsilon, v_\varepsilon^a)\right), \partial^\alpha \widetilde{v}_\varepsilon^R \right\rangle \\ &- \varepsilon^{R+M-2} \left\langle \partial^\alpha (\widetilde{v}_\varepsilon^{1R}\, \partial_\theta \widetilde{v}_\varepsilon^R + \varepsilon\, \widetilde{v}_\varepsilon^{2R}\, \partial_y \widetilde{v}_\varepsilon^R), \partial^\alpha \widetilde{v}_\varepsilon^R \right\rangle \\ &- C\, 2^{-1} \varepsilon^{2\nu - M - R - 2} \left\langle {}^t(\partial_\theta, \varepsilon\, \partial_y) \partial^\alpha \left(q_\varepsilon^a + \varepsilon^R\, \widetilde{q}_\varepsilon^R\right)^2, \partial^\alpha \widetilde{v}_\varepsilon^R \right\rangle .\end{aligned}$$

We estimate the three above contributions.

$\circ$ *The first term: contribution of the approximated solution.* v_ε^a is an approximated solution for the operator $\mathcal{L}^a$. According to Proposition 1.1, it satisfies Inequality (11). That is to say, Inequality (103) holds and:

$$\begin{aligned}\left| \varepsilon^{N-R} \left\langle \partial^\alpha \left(\varepsilon^{-N} \mathcal{L}^a(\varepsilon, v_\varepsilon^a)\right), \partial^\alpha \widetilde{v}_\varepsilon^R \right\rangle \right| &\le \frac{1}{2} \left(\varepsilon^{2(N-R)} \left\| \partial^\alpha \left(\varepsilon^{-N} \mathcal{L}^a(\varepsilon, v_\varepsilon^a)\right) \right\|_{L^2}^2 + \left\| \partial^\alpha v_\varepsilon^R \right\|_{L^2}^2 \right), \\ &\le \frac{1}{2} \left(C_{\mathcal{L}^a}\, \varepsilon^{2(N-R)} + \left\| v_\varepsilon^R \right\|_{\overset{\circ}{H}{}^{|\alpha|}}^2 \right). \qquad (112)\end{aligned}$$

$\circ$ *The second term* is the contribution corresponding to the non linear part "$u \cdot \nabla u$" (in Navier-Stokes Equations). Select a positive constant c_1. Further, we choose it so that the contribution of $\nabla \widetilde{v}_\varepsilon^R$ in H^m-norm is small with respect to Φ_ε.

$$\begin{aligned}&\varepsilon^{R+M-2} \left| \left\langle \partial^\alpha (\widetilde{v}_\varepsilon^{1R} \partial_\theta \widetilde{v}_\varepsilon^R + \varepsilon\, \widetilde{v}_\varepsilon^{2R}\, \partial_y \widetilde{v}_\varepsilon^R), \partial^\alpha \widetilde{v}_\varepsilon^R \right\rangle \right| \\ &\quad\le \frac{1}{2} \left(c_1 \varepsilon^{2(R+M-2)} \left\| \partial^\alpha \left(\widetilde{v}_\varepsilon^{1R} \partial_\theta \widetilde{v}_\varepsilon^R + \widetilde{v}_\varepsilon^{2R} \partial_y \widetilde{v}_\varepsilon^R \right) \right\|_{L^2}^2 + \frac{1}{c_1} \left\| \partial^\alpha v_\varepsilon^R \right\|_{L^2}^2 \right), \\ &\quad\le c_1 \varepsilon^{2(R+M-2)} \left(\left\| \partial^\alpha \left(\widetilde{v}_\varepsilon^{1R} \partial_\theta \widetilde{v}_\varepsilon^R \right) \right\|_{L^2}^2 + \left\| \partial^\alpha \left(\widetilde{v}_\varepsilon^{2R} \partial_y \widetilde{v}_\varepsilon^R \right) \right\|_{L^2}^2 \right) + \frac{1}{2c_1} \left\| \partial^\alpha \widetilde{v}_\varepsilon^{1R} \right\|_{L^2}^2 . \qquad (113)\end{aligned}$$

We deal with the nonlinear term $\partial^\alpha (\widetilde{v}_\varepsilon^{1R} \partial_\theta \widetilde{v}_\varepsilon^R)$ applying the Gagliardo-Niremberg's estimate together with (99). There exists a positive constant C_g which only depends on m such that for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned}\left\| \partial^\alpha \left(\widetilde{v}_\varepsilon^{1R}\, \partial_\theta \widetilde{v}_\varepsilon^R \right) \right\|_{L^2}^2 &\le C_g^2 \left(\left\| \widetilde{v}_\varepsilon^{1R} \right\|_{L^\infty} \left\| \partial_\theta \widetilde{v}_\varepsilon^R \right\|_{\overset{\circ}{H}{}^{|\alpha|}} + \left\| \widetilde{v}_\varepsilon^{1R} \right\|_{\overset{\circ}{H}{}^{|\alpha|}} \left\| \partial_\theta \widetilde{v}_\varepsilon^R \right\|_{L^\infty} \right)^2, \\ &\le 4\, C_g^2 \left(\left\| v_\varepsilon^R \right\|_{\overset{\circ}{H}{}^{|\alpha|+1}} + \left\| \widetilde{v}_\varepsilon^R \right\|_{\overset{\circ}{H}{}^{|\alpha|}} \right)^2 \le 8\, C_g^2 \left(\left\| \widetilde{v}_\varepsilon^R \right\|_{\overset{\circ}{H}{}^{|\alpha|+1}}^2 + \left\| \widetilde{v}_\varepsilon^R \right\|_{\overset{\circ}{H}{}^{|\alpha|}}^2 \right).\end{aligned}$$

The same holds for $\widetilde{v}_\varepsilon^{2R}\, \partial_y \widetilde{v}_\varepsilon^R$. Finally, we get that for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$

$$\begin{aligned}&\left| \varepsilon^{R+M-2} \left\langle \partial^\alpha (\widetilde{v}_\varepsilon^{1R}\, \partial_\theta \widetilde{v}_\varepsilon^R + \varepsilon\, \widetilde{v}_\varepsilon^{2R}\, \partial_y \widetilde{v}_\varepsilon^R), \partial^\alpha \widetilde{v}_\varepsilon^R \right\rangle \right| \\ &\qquad\qquad \le 16\, c_1\, C_g^2 \left\| \widetilde{v}_\varepsilon^R \right\|_{\overset{\circ}{H}{}^{|\alpha|+1}}^2 + (16\, c_1\, C_g^2 + \frac{1}{2c_1}) \left\| \widetilde{v}_\varepsilon^R \right\|_{\overset{\circ}{H}{}^{|\alpha|}}^2 . \qquad (114)\end{aligned}$$

$\circ$ *The last term.* This term has to be studied with care. As mentioned, two obstacles have to be overcome:

-It is singular with respect to the number of derivatives acting on q. However the term can formally be written under the form:

$$\frac{1}{2} \left\langle \nabla(q^2), v \right\rangle .$$

A derivative can be passed onto the velocity with an integration by parts.

-The pressure is only estimated in anisotropic Sobolev spaces $H^m_{(1,\varepsilon)}$. We pass to the anisotropic Sobolev spaces $H^m_{(1,\varepsilon)}$ thanks to Inequality (92) at a cost on the precision w_m (see Equation (18)).

First we integrate by parts:

$$\left|-C\,2^{-1}\varepsilon^{2\nu-M-R-2}\,\left\langle {}^t(\partial_\theta,\varepsilon\,\partial_y)\partial^\alpha\left(q^a_\varepsilon+\varepsilon^R\,\widetilde{q}^R_\varepsilon\right)^2,\partial^\alpha\widetilde{v}^R_\varepsilon\right\rangle\right| = \left|C\,2^{-1}\varepsilon^{2\nu-M-R-2}\,\left\langle \partial^\alpha\left(q^a_\varepsilon+\varepsilon^R\,\widetilde{q}^R_\varepsilon\right)^2,\partial^\alpha\,{}^t(\partial_\theta,\varepsilon\,\partial_y)\widetilde{v}^R_\varepsilon\right\rangle\right|.$$

We select a positive constant c_2, then

$$\left|-C\,2^{-1}\varepsilon^{2\nu-M-R-2}\,\left\langle {}^t(\partial_\theta,\varepsilon\,\partial_y)\partial^\alpha\left(q^a_\varepsilon+\varepsilon^R\,\widetilde{q}^R_\varepsilon\right)^2,\partial^\alpha\widetilde{v}^R_\varepsilon\right\rangle\right| \leq \frac{C^2\varepsilon^{2(2\nu-R-M-2)}}{8c_2}\left\|\partial^\alpha\left(q^a_\varepsilon+\varepsilon^R\widetilde{q}^R_\varepsilon\right)^2\right\|^2_{L^2}+\frac{c_2}{2}\left\|\partial^\alpha\,{}^t(\partial_\theta,\varepsilon\,\partial_y)v^R_\varepsilon\right\|^2_{L^2}. \tag{115}$$

We apply the Gagliardo-Niremberg inequality together with the equivalence of norms (92),

$$\begin{aligned}\left\|\partial^\alpha\left(q^a_\varepsilon+\varepsilon^R\widetilde{q}^R_\varepsilon\right)^2\right\|_{L^2} &\leq 2C_g\left\|q^a_\varepsilon+\varepsilon^R\widetilde{q}^R_\varepsilon\right\|_{L^\infty}\left\|q^a_\varepsilon+\varepsilon^R\widetilde{q}^R_\varepsilon\right\|_{\overset{\circ}{H}{}^{|\alpha|}},\\ &\leq 2C_g\varepsilon^{-|\alpha|}\left\|q^a_\varepsilon+\varepsilon^R\widetilde{q}^R_\varepsilon\right\|_{L^\infty}\left\|q^a_\varepsilon+\varepsilon^R\widetilde{q}^R_\varepsilon\right\|_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}}.\end{aligned}$$

With regards to the construction of the characterized time T^*_ε, (99) and (102) holds. It results in:

$$\forall\,\varepsilon\in]0,1],\qquad\forall\,t\in[0,T^*_\varepsilon],\qquad\left\|\sqrt{\varepsilon}\,\widetilde{q}^R_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{T}\times\mathbb{R})}\leq 2,\quad\left\|\sqrt{\varepsilon}\,q^a_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{T}\times\mathbb{R})}\leq C_a\,.$$

We obtain

$$\left\|\partial^\alpha\left(q^a_\varepsilon+\varepsilon^R\widetilde{q}^R_\varepsilon\right)^2\right\|_{L^2}\leq 2\,C_g\,\varepsilon^{-1/2-|\alpha|}(C_a+2)^2. \tag{116}$$

Finally plugging Inequality (116) in (115), we deduce:

$$\left|-C\,2^{-1}\varepsilon^{2\nu-M-R-2}\,\left\langle {}^t(\partial_\theta,\varepsilon\,\partial_y)\partial^\alpha\left(q^a_\varepsilon+\varepsilon^R\,\widetilde{q}^R_\varepsilon\right)^2,\partial^\alpha\widetilde{v}^R_\varepsilon\right\rangle\right| \leq \frac{C^2\,C_g^2\,\varepsilon^{2(2\nu-R-M-5/2-|\alpha|)}}{2c_2}(C_a+2)^4+\frac{c_2}{2}\left\|v^R_\varepsilon\right\|^2_{\overset{\circ}{H}{}^{|\alpha|+1}}. \tag{117}$$

$\circ$ *To finish*, we put estimates (112), (114) and (117) together. Let c_1 and c_2 be two positive constants for all $\varepsilon\in]0,1]$ and for all time $t\in[0,T^*_\varepsilon]$:

$$\begin{aligned}|\langle\partial^\alpha S^{R,N}_\varepsilon,\partial^\alpha v^R_\varepsilon\rangle| &\leq \left(\frac{c_2}{2}+16\,c_1\,C_g^2\right)\left\|v^R_\varepsilon\right\|^2_{\overset{\circ}{H}{}^{|\alpha|+1}}+\left(\frac{1}{2}+16\,c_1\,C_g^2+\frac{1}{2c_1}\right)\left\|v^R_\varepsilon\right\|^2_{\overset{\circ}{H}{}^{|\alpha|}}\\ &\quad+\left(C_{\mathcal{L}^a}\,\varepsilon^{2(N-R)}+\frac{C^2\,C_g^2}{2c_2}(C_a+2)^4\,\varepsilon^{2(2\mathcal{V}-R-M-5/2-|\alpha|)}\right),\\ &\leq \left(\frac{c_2}{2}+16\,c_1\,C_g^2\right)\left\|v^R_\varepsilon\right\|^2_{\overset{\circ}{H}{}^{|\alpha|+1}}+\left(\frac{1}{2}+16\,c_1\,C_g^2+\frac{1}{2c_1}\right)\left\|v^R_\varepsilon\right\|^2_{\overset{\circ}{H}{}^{|\alpha|}}\\ &\quad+\left(C_{\mathcal{L}^a}+\frac{C^2\,C_g^2}{2c_2}(C_a+2)^4\right)\varepsilon^{w_m}.\end{aligned}$$

Select C_S a positive constant. We choose c_1 and c_2 two positive constants such that $C_S := \frac{c_2}{2}+16\,c_1\,C_g^2$. Then, it requires $C_S^1=\frac{1}{2}+16\,c_1\,C_g^2+\frac{1}{2c_1}$ and $C_p=C_{\mathcal{L}^a}+\frac{C\,C_g^2}{2\,c_2}(C_a+2)^4$. $\square$

Contribution of operators $\mathcal{H}$ and $\mathcal{C}$. Presently, we study the singular operators $\mathcal{H}$ and $\mathcal{C}$. We decompose their action into:

$$\begin{aligned}\left\langle \partial^\alpha \left(\mathcal{H}\widetilde{v}_\varepsilon^R\right), \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle &= \left\langle \mathcal{H}\left(\partial^\alpha \widetilde{v}_\varepsilon^R\right), \widetilde{v}_\varepsilon^R\right\rangle + \left\langle [\partial^\alpha, \mathcal{H}]\widetilde{v}_\varepsilon^R, \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle ,\\ \left\langle \partial^\alpha \left(\mathcal{C}\widetilde{v}_\varepsilon^R\right), \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle &= \left\langle \mathcal{C}\left(\partial^\alpha \widetilde{v}_\varepsilon^R\right), \widetilde{v}_\varepsilon^R\right\rangle + \left\langle [\partial^\alpha, \mathcal{C}]\widetilde{v}_\varepsilon^R, \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle .\end{aligned}$$

by making some commutators appear. Since the commutator of two operators respectively of order m and n is of order $m+n-1$, the contribution of the commutators are less singular than expected:

Lemma 3.7. *Select a multi-index $\alpha \in \mathbb{N}^2$ with length satisfying $1 \le |\alpha| \le m+3$. Select $\mathcal{V} \in \{\mathcal{H}, \mathcal{C}\}$. There exist $C_{\mathcal{V}}^1$, $C_{\mathcal{V}}^2$ and $C_{\mathcal{V}}^3$ three positive constants such that for all $\varepsilon \in]0,1]$, for all time $t \in [0, T_\varepsilon^*]$,*

$$\begin{aligned}\left|\left\langle [\partial^\alpha, \mathcal{V}]\,\widetilde{v}_\varepsilon^R, \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle\right|(t) \le C_{\mathcal{V}}^1 \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{|\alpha|}(\mathbb{T}\times\mathbb{R})} &+ C_{\mathcal{V}}^2 \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^{|\alpha|-1}(\mathbb{T}\times\mathbb{R})}\\ &+ C_{\mathcal{V}}^3 \left\|\varepsilon^{-(1-\delta_{\alpha_1,0})}\partial^\alpha \widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})} .\end{aligned}$$

The proof is rather easy. We do not write it. Then, since $\left\langle \mathcal{H}\,\partial^\alpha \widetilde{v}_\varepsilon^R, \widetilde{v}_\varepsilon^R\right\rangle = 0$, there remains to deal with $\left\langle \mathcal{C}(\partial^\alpha \widetilde{v}_\varepsilon^R), \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle$.

Lemma 3.8. *[Absorption of $\mathcal{C}$] Select a positive constant c_1 and a multi-index $\alpha \in \mathbb{N}^2$ of length less than $m+3$. Then for all $\varepsilon \in]0,1]$ for all time $t \in [0, T_\varepsilon^*]$:*

$$\langle \mathcal{C}\partial^\alpha \widetilde{v}_\varepsilon^R(t,\cdot), \partial^\alpha \widetilde{v}_\varepsilon^R(t,\cdot)\rangle \le \frac{\|h\|_{L^\infty(\mathbb{T})}}{2c_1} \|\partial^\alpha \widetilde{v}_\varepsilon^R(t,\cdot)\|^2_{L^2(\mathbb{T}\times\mathbb{R})} + \frac{c_1}{2}\|h\|_{L^\infty} \left\|\varepsilon^{-1}\partial_\theta \partial^\alpha \widetilde{v}_\varepsilon^R\right\|^2_{L^2} .$$

Proof of Lemma 3.8. We prove the result for $\alpha = (0,0)$. Replacing $\widetilde{v}_\varepsilon^R$ by $\partial^\alpha \widetilde{v}_\varepsilon^R$ in the above estimates proves the result. We select c_1 a positive constant. The idea is to integrate by parts with respect to the variable θ to make the weighted derivative of the velocity $\varepsilon^{-1}\partial_\theta \widetilde{v}_\varepsilon^R$ appear.

$$\begin{aligned}|\langle \mathcal{C}\widetilde{v}_\varepsilon^R, \widetilde{v}_\varepsilon^R\rangle| &= \left|\int h\,\varepsilon^{-1}\partial_\theta \widetilde{v}_\varepsilon^{1R}\,\widetilde{v}_\varepsilon^{2R} d\theta\, dy + \int h\,\widetilde{v}_\varepsilon^{1R}\,\varepsilon^{-1}\partial_\theta \widetilde{v}_\varepsilon^{2R} d\theta\, dy\right| ,\\ &\le \frac{\|h\|_{L^\infty}}{2}\left(\frac{1}{c_1}\left\|\widetilde{v}_\varepsilon^R\right\|^2_{L^2} + c_1 \left\|\varepsilon^{-1}\partial_\theta \widetilde{v}_\varepsilon^R\right\|^2_{L^2}\right) .\end{aligned}$$

□

Absorption of singular terms. Select c_1 and C_s two positive constants to be chosen (small) later. We get a bound for the right-hand-side of (111) applying Lemmas 3.6-...-3.8. We obtain that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned}\frac{1}{2}\partial_t \left\|\partial^\alpha \widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{L^2} + \boxed{c_0\,\Phi_\varepsilon\left(\nabla, \partial^\alpha \widetilde{v}_\varepsilon^R\right) - \tfrac{c_1}{2}\|h\|_{L^\infty}\left\|\varepsilon^{-1}\partial_\theta \partial^\alpha \widetilde{v}_\varepsilon^R\right\|^2_{L^2} - C_S \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{J+1}(\mathbb{T}\times\mathbb{R})}}\\ \le \left(\frac{\|h\|_{L^\infty}}{2c_1} + C_S^1 + C_{\mathcal{C}}^1 + C_{\mathcal{H}}^1\right)\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{J}(\mathbb{T}\times\mathbb{R})} + C_p\,\varepsilon^{2w_m}\\ + (C_{\mathcal{C}}^2 + C_{\mathcal{H}}^2)\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^{J-1}(\mathbb{T}\times\mathbb{R})} + (C_{\mathcal{C}}^3 + C_{\mathcal{H}}^3)\left\|\varepsilon^{-(1-\delta_{\alpha_1,0})}\partial^\alpha \widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})}\\ + \left|\left\langle \partial^\alpha \left(\mathcal{A}\widetilde{v}_\varepsilon^R\right), \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle\right| + \left|\left\langle \partial^\alpha \left(\mathcal{B}\widetilde{v}_\varepsilon^R\right), \partial^\alpha \widetilde{v}_\varepsilon^R\right\rangle\right| . \quad (118)\end{aligned}$$

At this stage, the quadratic form Φ_ε can not absorb the norm $\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{J+1}(\mathbb{T}\times\mathbb{R})}$. However summing over all $\alpha \in \mathbb{N}^2$ of length J the quadratic form $\sum_{|\alpha|=J} \Phi_\varepsilon(\nabla, \partial^\alpha \widetilde{v}_\varepsilon^R)$ does.

Choosing c_1 and C_S^1 such that the quantity $\widetilde{c}_0 := c_0 - \frac{c_1}{2}\left\|h\right\|_{L^\infty} - (m+4)C_S^1$ is positive, we have:

$$\begin{aligned}
&\sum_{|\alpha|=J}\left(c_0\,\Phi_\varepsilon\left(\nabla,\partial^\alpha\widetilde{v}_\varepsilon^R\right) - \frac{c_1}{2}\left\|h\right\|_{L^\infty}\left\|\varepsilon^{-1}\partial_\theta\partial^\alpha\widetilde{v}_\varepsilon^R\right\|^2_{L^2} - C_S\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{J+1}(\mathbb{T}\times\mathbb{R})}\right)\\
&\qquad\geq c_0\sum_{|\alpha|=J}\Phi_\varepsilon(\nabla,\partial^\alpha\widetilde{v}_\varepsilon^R) - \frac{c_1}{2}\left\|h\right\|_{L^\infty}\sum_{|\alpha|=J}\left\|\varepsilon^{-1}\partial_\theta\partial^\alpha\widetilde{v}_\varepsilon^R\right\|^2_{L^2} - (J+1)C_S\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{J+1}(\mathbb{T}\times\mathbb{R})}\\
&\qquad\geq\left(c_0 - \frac{c_1}{2}\left\|h\right\|_{L^\infty} - (m+4)C_S\right)\sum_{|\alpha|=J}\Phi_\varepsilon(\nabla,\partial^\alpha\widetilde{v}_\varepsilon^R) = \widetilde{c}_0\sum_{|\alpha|=J}\Phi_\varepsilon(\nabla,\partial^\alpha\widetilde{v}_\varepsilon^R).
\end{aligned}$$

Hence, we obtain for all $\varepsilon\in]0,\varepsilon_d]$ and for all time $t\in[0,T_\varepsilon^*]$:

$$\begin{aligned}
&\frac{1}{2}\partial_t\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{J}(\mathbb{T}\times\mathbb{R})} + \widetilde{c}_0\sum_{|\alpha|=J}\Phi_\varepsilon\left(\nabla,\partial^\alpha\widetilde{v}_\varepsilon^R\right)\\
&\qquad\leq (J+1)\left(\frac{\left\|h\right\|_{L^\infty}}{2c_1} + C_S^1 + C_{\mathcal{C}}^1 + C_{\mathcal{H}}^1\right)\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{J}(\mathbb{T}\times\mathbb{R})} + C_p(J+1)\,\varepsilon^{2w_m}\\
&\quad + (J+1)(C_{\mathcal{C}}^2 + C_{\mathcal{H}}^2)\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^{J-1}(\mathbb{T}\times\mathbb{R})} + (C_{\mathcal{C}}^3 + C_{\mathcal{H}}^3)\sum_{|\alpha|=J}\left\|\varepsilon^{-(1-\delta_{\alpha_1,0})}\partial^\alpha\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})}\\
&\qquad\qquad + \sum_{|\alpha|=J}\left|\left\langle\partial^\alpha\left(\mathcal{A}\widetilde{v}_\varepsilon^R\right),\partial^\alpha\widetilde{v}_\varepsilon^R\right\rangle\right| + \sum_{|\alpha|=J}\left|\left\langle\partial^\alpha\left(\mathcal{B}\widetilde{v}_\varepsilon^R\right),\partial^\alpha\widetilde{v}_\varepsilon^R\right\rangle\right|. \qquad (119)
\end{aligned}$$

Remark 3. *The term $\sum_{|\alpha|=J}\left\|\varepsilon^{-(1-\delta_{\alpha_1,0})}\partial^\alpha\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})}$ seems to be singular with respect to ε. With regards to the Inequality (101), we will interpret it as a non-degenerate source term in the proof of Proposition (3.1).*

3.1.3 Last step: estimates over non-singular terms

The two last terms (contribution of operator $\mathcal{A}$ and operator $\mathcal{B}$) can be easily estimated. They satisfy:

Lemma 3.9. *Select a multi-index $\alpha\in\mathbb{N}^2$ of length smaller than $m+3$. Select $\mathcal{V}\in\{\mathcal{A},\mathcal{B}\}$. There exist $C_{\mathcal{V}}^1$ and $C_{\mathcal{V}}^2$ two positive constants such that for all $\varepsilon\in]0,1]$ and for all time $t\in[0,T_\varepsilon^*]$:*

$$\left|\left\langle\partial^\alpha\left(\mathcal{V}\widetilde{v}_\varepsilon^R\right),\partial^\alpha\widetilde{v}_\varepsilon^R\right\rangle\right|(t) \leq C_{\mathcal{V}}^1\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{|\alpha|}(\mathbb{T}\times\mathbb{R})} + C_{\mathcal{V}}^2\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^{|\alpha|-1}(\mathbb{T}\times\mathbb{R})}. \qquad (120)$$

The proof is very classical. We do not write it. Applying Lemma 3.9, we obtain from Inequality (119) that for all $\varepsilon\in]0,\varepsilon_d]$ and for all time $t\in[0,T_\varepsilon^*]$:

$$\begin{aligned}
&\frac{1}{2}\partial_t\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{J}(\mathbb{T}\times\mathbb{R})} + \widetilde{c}_0\sum_{|\alpha|=J}\Phi_\varepsilon\left(\nabla,\partial^\alpha\widetilde{v}_\varepsilon^R\right) \leq C_J^1\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{J}(\mathbb{T}\times\mathbb{R})} + C_p(J+1)\,\varepsilon^{2w_m}\\
&\qquad + C_J^2\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^{J-1}(\mathbb{T}\times\mathbb{R})} + C_J^3\sum_{|\alpha|=J}\left\|\varepsilon^{-(1-\delta_{\alpha_1,0})}\partial^\alpha\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})}, \qquad (121)
\end{aligned}$$

with $C_J^1 := (J+1)\left(\frac{\|h\|_{L^\infty}}{2c_1} + C_S^1 + C_{\mathcal{A}}^1 + C_{\mathcal{B}}^1 + C_{\mathcal{C}}^1 + C_{\mathcal{H}}^1\right)$, $C_J^2 := (J+1)(C_{\mathcal{A}}^2 + C_{\mathcal{B}}^2 + C_{\mathcal{C}}^2 + C_{\mathcal{H}}^2)$ and $C_J^3 := (C_{\mathcal{C}}^3 + C_{\mathcal{H}}^3)$.

That achieves the proof of Lemma 3.4. □

3.1.4 Proof of Proposition 3.1

This subsection is dedicated to the proof of Proposition 3.1. We prove by induction that property $\mathcal{P}(J)$, defined at the level of (105), is satisfied for $J \in [\![0, m+3]\!]$. We proceed in two steps. First we obtain an estimate for the velocity (Inequality (100)). Then, we take advantage of the quadratic form $\sum_{|\alpha|=J} \Phi_\varepsilon\left(\nabla, \partial^\alpha \widetilde{v}_\varepsilon^R\right)$ to obtain the regularization Inequality (101).

Since the proof is exactly the same for the case $J=0$ and the increment of the induction, we do not write the details to prove that property $\mathcal{P}(0)$ is true.

We assume that $\mathcal{P}(J)$ is true for $J \in [\![0, m+2]\!]$. We apply Lemma in the case $J+1$ (≥ 1). There exist ε_d and c_1 two positive constants, there exist C_{J+1}^1, C_{J+1}^2, C_{J+1}^3 and C_p four positive constants such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned}\frac{1}{2}\partial_t \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\mathring{H}^{J+1}(\mathbb{T}\times\mathbb{R})} + c_1 \sum_{|\alpha|=J+1} \Phi_\varepsilon(\nabla, \partial^\alpha \widetilde{v}_\varepsilon^R)(t) \\ \leq C_{J+1}^1 \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^J(\mathbb{T}\times\mathbb{R})} + C_{J+1}^2 \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\mathring{H}^{J+1}(\mathbb{T}\times\mathbb{R})} \\ + C_{J+1}^3 \sum_{|\alpha|=J+1} \left\|\varepsilon^{-(1-\delta_{\alpha_1,0})}\partial^\alpha \widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})} + (J+2)\, C_p\, \varepsilon^{2w_m}. \end{aligned} \tag{122}$$

∘ *First*, we look for the H^{J+1}-estimate. We neglect $\sum_{|\alpha|=J+1} \Phi_\varepsilon(\nabla, \partial^\alpha \widetilde{v}_\varepsilon^R)$ since it is positive as a sum of positive quadratic forms. Hence, for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\frac{1}{2}\partial_t \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\mathring{H}^{J+1}(\mathbb{T}\times\mathbb{R})} \leq C_{J+1}^2 \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{\mathring{H}^{J+1}(\mathbb{T}\times\mathbb{R})} + S_{J+1}(t)\,,$$

where S_{J+1} is defined as:

$$\begin{aligned} S_{J+1}(t) := C_{J+1}^1 \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^J(\mathbb{T}\times\mathbb{R})} \\ + C_{J+1}^3 \sum_{|\alpha|=J+1} \left\|\varepsilon^{-(1-\delta_{\alpha_1,0})}\partial^\alpha \widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})} + (J+2)\, C_p\, \varepsilon^{2w_m}. \end{aligned}$$

From the assumption $\mathcal{P}(J)$, Inequalities (100) and (101) hold. The functions $t \longmapsto \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^J}$ and $t \longmapsto \sum_{|\alpha|=J+1} \left\|\varepsilon^{-(1-\delta_{\alpha_1,0})}\partial^\alpha \widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})}$ are in $L^1([0, T_\varepsilon^*])$. In addition there exists M_m^1 a positive constant such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^J} \leq M_m^1\, \varepsilon^{2w_m} \qquad \text{and} \qquad \sum_{|\alpha|=J+1} \left\|\varepsilon^{-(1-\delta_{\alpha_1,0})}\partial^\alpha \widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{L^2(\mathbb{T}\times\mathbb{R})} \leq M_m^1\, \varepsilon^{2w_m}.$$

Thus, the function $t \longmapsto S_{J+1}(t)$ is in $L^1([0,T_\varepsilon^*])$ and satisfies for all $\varepsilon \in]0,\varepsilon_d]$ and for all time $t \in [0,T_\varepsilon^*]$:

$$S_{J+1}(t) \leq \left(2M_m^1 + (J+2)C_p\right)\varepsilon^{2w_m}.$$

We can apply the Gronwall's lemma, for all $\varepsilon \in]0,\varepsilon_d]$ and for all time $t \in [0,T_\varepsilon^*]$,

$$\|\widetilde{v}_\varepsilon^R(t,\cdot)\|^2_{\overset{\circ}{H}{}^{J+1}(\mathbb{T}\times\mathbb{R})} \leq \int_0^t e^{2C_{J+1}^2(t-s)} S_{J+1}(s)ds \leq \frac{2M_m^1 + (J+2)C_p}{2C_{J+1}^2}\varepsilon^{2w_m}\left(e^{2C_{J+1}^2 t} - 1\right). \quad (123)$$

Setting $K_{J+1}^1 := (2M_m^1 + (J+2)C_p)/(2C_{J+1}^2)$ and $K_{j+1}^2 := 2C_{J+1}^2$, the first inequality of $\mathcal{P}(J+1)$ is proved.

$\circ$ *Then*, to obtain the estimation $L^2([0,T_\varepsilon^*],H^{J+2})$, we go back to Equation (122). We integrate it with respect to the time t,

$$\frac{1}{2}\|\widetilde{v}_\varepsilon^R(t,.)\|^2_{\overset{\circ}{H}{}^{J+1}} + \int_0^t \sum_{|\alpha|=J+1} \Phi_\varepsilon(\nabla,\partial^\alpha\widetilde{v}_\varepsilon^R)(s)ds$$
$$\leq C_{J+1}^2 \int_0^t \left(\|\widetilde{v}_\varepsilon^R(s,\cdot)\|^2_{\overset{\circ}{H}{}^{J+1}} + S_{J+1}(s)\right)ds. \quad (124)$$

Since $\|\widetilde{v}_\varepsilon^R(t,.)\|^2_{\overset{\circ}{H}{}^{J+1}}$ is positive, we neglect it in the left-hand side of the Inequality (124). According to (123), the functions $t \longmapsto \|\widetilde{v}_\varepsilon^R(t,\cdot)\|^2_{\overset{\circ}{H}{}^{J+1}}$ and S_{J+1} are in $L^1([0,T_\varepsilon^*])$. Furthermore, there exists K_{J+1}^2 such that for all $\varepsilon \in]0,\varepsilon_d]$ and for all time $t \in [0,T_\varepsilon^*]$:

$$\|\widetilde{v}_\varepsilon^R(t,\cdot)\|^2_{\overset{\circ}{H}{}^{J+1}} + S(t) \leq K_{J+1}^2\varepsilon^{2w_m}\, t \leq K_{J+1}^2\varepsilon^{2w_m}.$$

So to say, there exists $K_{J+1}^2(>0)$ such that for all $\varepsilon \in]0,\varepsilon_d]$ and for all time $t \in [0,T_\varepsilon^*]$:

$$\int_0^t \sum_{|\alpha|=J+1} \Phi_\varepsilon(\nabla,\partial^\alpha\widetilde{v}_\varepsilon^R)(s)ds \leq \varepsilon^{2w_s}\, K_{J+1}^2\, t.$$

We end the proof choosing $K_m^i := \max_{j\in[\![0,m+4]\!]} K_j^i$ $(i \in \{1,2\})$. $\square$

Remark 4. *In the above discussion, we only use the assumption $M \geq 2$ whereas we assume $M \geq 7/2$. It becomes crucial when estimating the pressure.*

3.2 Control over the pressure

In Subsection 3.2 we prove Proposition 3.2. The result is again proved by induction on the size m setting $\mathcal{Q}(m)$:

$$\mathcal{Q}(m) := \text{" The Proposition 3.2 is satisfied up to the integer } m\text{".} \quad (125)$$

To go from m to $m+1$, the proof is once more based on an energy method for Equation (94a) in the anisotropic Sobolev spaces $H^m_{(1,\varepsilon)}$ (defined page 4):

Lemma 3.10. *There exits a positive constant $\varepsilon_d > 0$ such that for any $J \in [\![0, m+3]\!]$, there exist three positive constants C_J^1, C_J^2 and C_J^3 such that for all $\varepsilon \in]0, \varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:*

$$\begin{aligned}\frac{1}{2}\partial_t \left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^J_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} \leq C_J^1 \left(1 + \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^{m+4}(\mathbb{T}\times\mathbb{R})}\right) \left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^J_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} \\ + C_J^2 \left(1 + \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^{m+4}}\right) \left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{H^{J-1}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} \\ + C_J^3 \left(\left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^{m+4}(\mathbb{T}\times\mathbb{R})} + \varepsilon^{2(N-R)}\right). \quad (126)\end{aligned}$$

The lack of dissipation has two consequences.

-There is no *absorption phenomena*. It makes us consider the anisotropic Sobolev spaces instead of the classical Sobolev spaces. An unexpected effect of considering the anisotropic Sobolev norms is that the family $\{\partial_\theta \widetilde{q}_\varepsilon^R\}_\varepsilon$ becomes singular with respect to ε in L^2. However $\{\varepsilon\partial_\theta \widetilde{q}_\varepsilon^R\}_\varepsilon$ is. We introduce a power of ε when necessary thanks to the integer M. It explains why M is assumed at least larger than 3.

-There is no *regularization phenomena* over the pressure. A difficulty appears when we want to estimate terms such as ∇p. Each time it appears, we integrate by parts (if possible) to pass the derivative on the velocity. It explains the appearance of the H^{m+4} norm of the velocity. The regularization of the velocity plays again a crucial role in this process.

The energy method in $H^m_{(1,\varepsilon)}$ consists in differentiating Equation (94b) by $\varepsilon^{\alpha_1}\partial^\alpha$. Then we multiply it by $\varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}_\varepsilon^R$ and integrate (with respect to the space variables (θ, y)),

$$\begin{aligned}\frac{1}{2}\partial_t \left\|\varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}_\varepsilon^R\right\|^2_{L^2} + \left\langle \varepsilon^{\alpha_1}\partial^\alpha(\mathcal{H}\widetilde{q}_\varepsilon^R), \varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}_\varepsilon^R\right\rangle \\ + \left\langle \varepsilon^{\alpha_1}\partial^\alpha(\mathcal{B}\widetilde{q}_\varepsilon^R), \varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}_\varepsilon^R\right\rangle + \left\langle \varepsilon^{\alpha_1}\partial^\alpha(\mathcal{D}\widetilde{v}_\varepsilon^R), \varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}_\varepsilon^R\right\rangle \\ = \left\langle \varepsilon^{\alpha_1}\partial^\alpha S_\varepsilon^{0,R,N}, \varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}_\varepsilon^R\right\rangle - \left\langle \varepsilon^{\alpha_1}\partial^\alpha(\mathcal{F}\widetilde{v}_\varepsilon^R), \varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}_\varepsilon^R\right\rangle, \quad (127)\end{aligned}$$

where operators $\mathcal{H}$, $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{F}$ are defined as

$$\begin{aligned}&\mathcal{H}\widetilde{q}_\varepsilon^R := \varepsilon^{-1} h\, \partial_y \widetilde{q}_\varepsilon^R, \quad \mathcal{F}\widetilde{v}_\varepsilon^R := \varepsilon^{M-2}\left(\widetilde{v}_\varepsilon^{1R}\,\partial_\theta q_\varepsilon^a + \varepsilon\, \widetilde{v}_\varepsilon^{2R}\,\partial_y q_\varepsilon^a\right) + C\,\varepsilon^{M-2} q_\varepsilon^a \left(\partial_\theta \widetilde{v}_\varepsilon^{1R} + \varepsilon\,\partial_y \widetilde{v}_\varepsilon^{2R}\right), \\ &\mathcal{B}\widetilde{q}_\varepsilon^R := \varepsilon^{M-2}\left(v_\varepsilon^{1a}\,\partial_\theta \widetilde{q}_\varepsilon^R + \varepsilon\, v_\varepsilon^{2a}\,\partial_y \widetilde{q}_\varepsilon^R\right), \quad \mathcal{D}\widetilde{q}_\varepsilon^R := C\,\varepsilon^{M-2}\widetilde{q}_\varepsilon^R\left(\partial_\theta v_\varepsilon^{1a} + \varepsilon\,\partial_y v_\varepsilon^{2a}\right).\end{aligned}$$

In the sequel we prove several lemmas where we estimate each contributions. We sort them contingent on how much they are singular with respect to ε in the anisotropic Sobolev spaces. In Subsection 3.2.1, we study the contribution of operators $\mathcal{H}$, $\mathcal{B}$ and $\mathcal{D}$. Then in Subsection 3.2.2, due to technical computations, we have to assume that $M \geq 7/2$ and deal with the terms $\mathcal{F}$ and $S_\varepsilon^{0,R,N}$.
Finally in Subsection 3.2.3, once Lemma 3.10 is proved, we present the induction to prove Proposition 3.2.

3.2.1 Non-singular terms: $M \geq 3$

First we deal with the contributions of operators $\mathcal{H}$, $\mathcal{B}$ and $\mathcal{D}$. Since we introduce the anisotropic Sobolev spaces to get ride of the singular transport $\mathcal{H}$, it is no longer singular:

Lemma 3.11. *Assume that $M \geq 2$. Select a multi-index $\alpha \in \mathbb{N}^2$ of length smaller than $m+3$. Then, there exist two positive constants $C_{\mathcal{H}}^1$ and $C_{\mathcal{H}}^2$ such that for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$:*

$$\left|\left\langle \varepsilon^{\alpha_1}\partial^\alpha\left(\varepsilon^{-1} h\,\partial_y \widetilde{q}_\varepsilon^R\right), \varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}_\varepsilon^R\right\rangle\right|(t) \leq C_{\mathcal{H}}^1 \left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} + C_{\mathcal{H}}^2 \left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{H^{|\alpha|-1}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})}.$$

When $|\alpha| = 0$, the contribution of $\mathcal{H}$ is even vanishing. Since it is very classical, we do not write the proof.

Presently we compute the contribution of operators $\mathcal{B}$ and $\mathcal{D}$. We have to estimate the term $\partial_\theta \widetilde{q}_\varepsilon^R$ singular in ε in the anisotropic spaces. It thus requires:

Lemma 3.12. *Assume $M \geq 3$. Select a multi-index $\alpha \in \mathbb{N}^2$ with length smaller than $m+3$. Select $\mathcal{V} \in \{\mathcal{B}, \mathcal{D}\}$. There exist two positive constants $C_{\mathcal{V}}^1$ and $C_{\mathcal{V}}^2$ such that for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$,*

$$\left| \left\langle \varepsilon^{\alpha_1} \partial^\alpha \left(\mathcal{V} \widetilde{q}_\varepsilon^R \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle \right| (t) \leq C_{\mathcal{V}}^1 \left\| \widetilde{q}_\varepsilon^R(t,\cdot) \right\|^2_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} + C_{\mathcal{V}}^2 \left\| \widetilde{q}_\varepsilon^R(t,\cdot) \right\|^2_{H^{|\alpha|-1}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} .$$

Of course the contribution of $\mathcal{D}$ only requires $M \geq 2$ since $\partial_y q_\varepsilon^R$ is bounded in the anisotropic Sobolev spaces.

Proof of Lemma 3.12. We prove the result for operator $\mathcal{B}$. We consider a multi-index $\alpha \in \mathbb{N}^2$ such that $|\alpha| \geq 1$. We have:

$$\begin{aligned}\left\langle \varepsilon^{\alpha_1} \partial^\alpha \left(\mathcal{B} \widetilde{q}_\varepsilon^R \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle = \varepsilon^{M-2} & \left\langle \varepsilon^{\alpha_1} \partial^\alpha \left(v_\varepsilon^{a1}\, \partial_\theta \widetilde{q}_\varepsilon^R \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle \\ & + \varepsilon^{M-2} \left\langle \varepsilon^{\alpha_1} \partial^\alpha \left(\varepsilon\, v_\varepsilon^{a2}\, \partial_y \widetilde{q}_\varepsilon^R \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle .\end{aligned}$$

The two terms are estimated performing the same proof. Thus we only write the proof for the term $\varepsilon^{M-2} \left\langle \varepsilon^{\alpha_1} \partial^\alpha \left(v_\varepsilon^{a1}\, \partial_\theta \widetilde{q}_\varepsilon^R \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right) \rangle$. Of course, the estimates for the second term only requires $M \geq 2$ since it appears the derivative ∂_y instead of ∂_θ.

We apply the Leibniz formula. Then to diminish the number of derivatives acting on the pressure (for the extremal term), we consider an integration by parts. There exists a family $\{C_{\alpha,\beta}\}$ of positive constants such that,

$$\begin{aligned}&\varepsilon^{M-2} \left| \left\langle \varepsilon^{\alpha_1} \partial^\alpha \left(v_\varepsilon^{a1}\, \partial_\theta \widetilde{q}_\varepsilon^R \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle \right| \\ &= \varepsilon^{M-2} \sum_{\beta<\alpha} C_{\alpha,\beta} \int_{\mathbb{T}\times\mathbb{R}} \varepsilon^{2\alpha_1} \partial^{\alpha-\beta} v_\varepsilon^{a1}\, \partial^\beta \left(\partial_\theta \widetilde{q}_\varepsilon^R \right) \partial^\alpha \widetilde{q}_\varepsilon^R d\theta\, dy \\ &\qquad\qquad - \varepsilon^{(M-2)}/2 \int_{\mathbb{T}\times\mathbb{R}} \partial_\theta v_\varepsilon^{a1} (\varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R)^2\, d\theta\, dy . \end{aligned} \tag{128}$$

$\circ$ *First*, we get a bound for the last term in the right-hand-side of Equation (128). For all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned}\left| -\varepsilon^{(M-2)}/2 \int_{\mathbb{T}\times\mathbb{R}} \partial_\theta v_\varepsilon^{a1}(t,\cdot) (\varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R(t,\cdot))^2\, d\theta\, dy \right| &\leq \| v_\varepsilon^a(t,\cdot) \|_{W^{1,\infty}} /2\, \| q_\varepsilon(t,\cdot) \|^2_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}} , \\ &\leq C_a/2\, \| q_\varepsilon(t,\cdot) \|^2_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}} . \end{aligned} \tag{129}$$

$\circ$ *Then*, to control the other terms in Equation (128), we make appear the ε-derivative $\varepsilon\, \partial_\theta$

paying a loss of precision on M. For all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned}
&\left| \varepsilon^{M-2} \int_{\mathbb{T}\times\mathbb{R}} \varepsilon^{2\alpha_1}\, \partial^{\alpha-\beta} v_\varepsilon^{a1}(t,\cdot)\, \partial^\beta \partial_\theta \widetilde{q}_\varepsilon^R(t,\cdot)\, \partial^\alpha \widetilde{q}_\varepsilon^R(t,\cdot) d\theta\, dy \right| \\
&\qquad = \left| \boxed{\varepsilon^{M-3}} \int_{\mathbb{T}\times\mathbb{R}} \varepsilon^{2\alpha_1+1}\, \partial^{\alpha-\beta} v_\varepsilon^{a1}(t,\cdot)\, \partial^\beta \partial_\theta \widetilde{q}_\varepsilon^R(t,\cdot)\, \partial^\alpha \widetilde{q}_\varepsilon^R(t,\cdot)\, d\theta\, dy \right| , \\
&\qquad \leq \frac{\left\| v_\varepsilon^{a1}(t,\cdot) \right\|_{W^{m+4,\infty}}}{2} \left(\left\| \varepsilon^{\alpha_1+1} \partial^\beta \partial_\theta \widetilde{q}_\varepsilon^R(t,\cdot) \right\|_{L^2}^2 + \left\| \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R(t,\cdot) \right\|_{L^2}^2 \right) , \\
&\qquad \leq \frac{C_a}{2} \left(\left\| \widetilde{q}_\varepsilon^R(t,\cdot) \right\|_{H^{|\alpha|-1}_{(1,\varepsilon)}}^2 + 2 \left\| \widetilde{q}_\varepsilon^R(t,\cdot) \right\|_{\mathring{H}^{|\alpha|}_{(1,\varepsilon)}}^2 \right) .
\end{aligned}$$

Choosing $A := \sum_{\beta<\alpha} C_{\alpha,\beta} C_a/2 > 0$ and $B := (\sum_{\beta<\alpha} C_{\alpha,\beta} + 1/2) C_a > 0$ we obtain that for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\varepsilon^{M-2} \left| \left\langle \varepsilon^{\alpha_1} \partial^\alpha \left(v_\varepsilon^{a1}\, \partial_\theta \widetilde{q}_\varepsilon^R \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle \right| (t) \leq A \left\| \widetilde{q}_\varepsilon^R(t,\cdot) \right\|_{H^{|\alpha|-1}_{(1,\varepsilon)}}^2 + B \left\| \widetilde{q}_\varepsilon^R(t,\cdot) \right\|_{\mathring{H}^{|\alpha|}_{(1,\varepsilon)}}^2 . \tag{130}$$

That ends the proof. □

3.2.2 Singular terms: $M \geq 7/2$

There remains to estimate the contributions of $\mathcal{F}$ and $S_\varepsilon^{0,R,N}$. They are put aside since we need $M \geq 7/2$ to desingularize it. It is actually a technical assumption and it should be relaxed to $M \geq 3$ by proving more regularity on the approximated solution $(q_\varepsilon^a, v_\varepsilon^a)$.

Estimate for operator $\mathcal{F}$. We start by estimating the contribution of the operator $\mathcal{F}$. The main difference with the three previous operators ($\mathcal{H}$, $\mathcal{B}$ and $\mathcal{D}$) is that the operator $\mathcal{F}$ acts on the velocity $\widetilde{v}_\varepsilon^R$. We have to be prudent with terms of the form

$$\partial^\alpha (q_\varepsilon^a\, (\partial_\theta \widetilde{v}_\varepsilon^{1R} + \partial_y \widetilde{v}_\varepsilon^{2R})).$$

First it is more singular with respect to the number of derivatives acting on the velocity. Once more, when estimating the product, we have to use a control over the pressure q_ε^a in L^∞-norm. We may lose some regularity to go back to the anisotropic Sobolev norms. This term can be more singular in L^2-norm. Thus it requires:

Lemma 3.13. *Assume $M \geq 7/2$. Select a multi-index $\alpha \in \mathbb{N}^2$ with length smaller than $m+3$. There exist two positive constants $C_{\mathcal{F}}^1$ and $S_{\mathcal{F}}$ such that for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$:*

$$\left| \left\langle \varepsilon^{\alpha_1} \partial^\alpha \left(\mathcal{F} \widetilde{v}_\varepsilon^R \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle \right| (t) \leq S_{\mathcal{F}} \left\| \widetilde{v}_\varepsilon^R(t,\cdot) \right\|_{H^{m+4}(\mathbb{T}\times\mathbb{R})}^2 + C_{\mathcal{F}}^1 \left\| \widetilde{q}_\varepsilon^R(t,\cdot) \right\|_{\mathring{H}^{|\alpha|}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})}^2 .$$

Proof of Lemma 3.13. Select a multi-index $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \leq m+3$. We decompose $\mathcal{F}$ as follows:

$$\begin{aligned}
\left\langle \varepsilon^{\alpha_1} \partial^\alpha \, (\mathcal{F} \widetilde{q}_\varepsilon^R), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle &= \left\langle \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha \left(\widetilde{v}_\varepsilon^{1R}\, \partial_\theta q_\varepsilon^a \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle \\
&\quad + \left\langle \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha \left(\varepsilon\, \widetilde{v}_\varepsilon^{2R}\, \partial_y q_\varepsilon^a \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle \\
+ \left\langle C\, \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha \left(q_\varepsilon^a\, \partial_\theta \widetilde{v}_\varepsilon^{1R} \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle &+ \left\langle C\, \varepsilon^{M-2} \varepsilon^{\alpha_1} \partial^\alpha \left(q_\varepsilon^a\, \varepsilon \partial_y \widetilde{v}_\varepsilon^{2R} \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle .
\end{aligned} \tag{131}$$

The first and second term (respectively the third and fourth term) can be computed following the same line. So we only provide the estimates for the first term (respectively the third term).

- *The first term.* We start by estimating the contribution of $\langle \varepsilon^{M-2}\varepsilon^{\alpha_1}\partial^\alpha \left(\widetilde{v}^{1R}_\varepsilon\, \partial_\theta q^a_\varepsilon\right), \varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}^R_\varepsilon\rangle$:

$$\langle \varepsilon^{M-2}\varepsilon^{\alpha_1}\partial^\alpha \left(\widetilde{v}^{1R}_\varepsilon\, \partial_\theta q^a_\varepsilon\right), \varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}^R_\varepsilon\rangle \leq \frac{1}{2}\left(\left\|\varepsilon^{M-2}\varepsilon^{\alpha_1}\partial^\alpha \left(\widetilde{v}^{1R}_\varepsilon\, \partial_\theta q^a_\varepsilon\right)\right\|^2_{L^2} + \left\|\varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}^R_\varepsilon\right\|^2_{L^2}\right)$$

We use the Leibniz formula. Then apply the Minkowski inequality and loss a power of ε to get a control of the approximated solution in the anisotropic Sobolev spaces. There exists a family of positive constant $\{C_{\alpha,\beta}\}$ such that

$$\begin{aligned}\left\|\varepsilon^{M-2}\varepsilon^{\alpha_1}\partial^\alpha \left(\widetilde{v}^{1R}_\varepsilon\, \partial_\theta q^a_\varepsilon\right)\right\|_{L^2} &\leq \varepsilon^{M-2}\sum_{0\leq\beta\leq\alpha} C_{\alpha,\beta}\left\|\varepsilon^{\alpha_1+1-1}\partial^\beta\partial_\theta q^a_\varepsilon\right\|_{L^\infty}\left\|\partial^{\alpha-\beta}\widetilde{v}^{1R}_\varepsilon\right\|_{L^2}\,,\\ &\leq \boxed{\varepsilon^{M-3}}\sum_{0\leq\beta\leq\alpha} C_{\alpha,\beta}\left\|q^a_\varepsilon\right\|_{W^{m+4,\infty}_{(1,\varepsilon)}}\left\|v^R_\varepsilon\right\|_{H^{m+3}}\,.\end{aligned}$$

Then, we use the imbeddings $\|q^a_\varepsilon(t,\cdot)\|_{W^{m+4,\infty}_{(1,\varepsilon)}} \leq \varepsilon^{-\frac{1}{2}}\|q^a_\varepsilon(t,\cdot)\|_{H^{m+6}_{(1,\varepsilon)}}$,

$$\begin{aligned}\left\|\varepsilon^{M-2}\varepsilon^{\alpha_1}\partial^\alpha \left(\widetilde{v}^{1R}_\varepsilon\, \partial_\theta q^a_\varepsilon\right)\right\|_{L^2} &\leq \boxed{\varepsilon^{M-7/2}}\sum_{0\leq\beta\leq\alpha} C_{\alpha,\beta}\left\|q^a_\varepsilon\right\|_{H^{m+6}_{(1,\varepsilon)}}\left\|v^R_\varepsilon\right\|_{H^{m+3}}\,,\\ &\leq C_a\sum_{0\leq\beta\leq\alpha} C_{\alpha,\beta}\left\|v^R_\varepsilon(t,\cdot)\right\|_{H^{m+3}}\,.\end{aligned}$$

Finally for all $\varepsilon\in]0,1]$ and for all time $t\in[0,T^*_\varepsilon]$,

$$\begin{aligned}&\left|\langle \varepsilon^{M-2}\varepsilon^{\alpha_1}\partial^\alpha \left(\widetilde{v}^{1R}_\varepsilon\, \partial_\theta q^a_\varepsilon\right), \varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}^R_\varepsilon\rangle\right|(t)\\ &\qquad\leq \frac{1}{2}\left(C_a^2\left(\sum_{0\leq\beta\leq\alpha} C_{\alpha,\beta}\right)^2\left\|v^R_\varepsilon(t,\cdot)\right\|^2_{H^{m+3}} + \left\|\varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}^R_\varepsilon(t,\cdot)\right\|^2_{L^2}\right)\,. \end{aligned}\tag{132}$$

- *Estimate of the third term.* We now move to the estimate of $\langle C\,\varepsilon^{M-2}\varepsilon^{\alpha_1}\partial^\alpha \left(q^a_\varepsilon\, \partial_\theta \widetilde{v}^{1R}_\varepsilon\right), \varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}^R_\varepsilon\rangle$.

First, apply the Cauchy-Schwarz inequality together with the Young inequality:

$$\left|\langle C\,\varepsilon^{M-2}\varepsilon^{\alpha_1}\partial^\alpha \left(q^a_\varepsilon\, \partial_\theta \widetilde{v}^{1R}_\varepsilon\right), \varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}^R_\varepsilon\rangle\right| \leq \frac{1}{2}\left(\left\|C\,\varepsilon^{M-2}\varepsilon^{\alpha_1}\partial^\alpha \left(q^a_\varepsilon\, \partial_\theta \widetilde{v}^{1R}_\varepsilon\right)\right\|^2_{L^2} + \left\|\varepsilon^{\alpha_1}\partial^\alpha \widetilde{q}^R_\varepsilon\right\|^2_{L^2}\right).$$

To get rid with the contribution of the nonlinear term $\left\|C\,\varepsilon^{M-2}\varepsilon^{\alpha_1}\partial^\alpha \left(q^a_\varepsilon\, \partial_\theta \widetilde{v}^{1R}_\varepsilon\right)\right\|_{L^2}$ we start with a Leibniz formula, then:

$$\begin{aligned}\left\|C\varepsilon^{M-2}\varepsilon^{\alpha_1}\partial^\alpha \left(q^a_\varepsilon\, \partial_\theta \widetilde{v}^{1R}_\varepsilon\right)\right\|_{L^2} &\leq\ C\varepsilon^{M-2}\sum_{0\leq\beta\leq\alpha} C_{\alpha,\beta}\left\|\varepsilon^{\alpha_1}\partial^{\alpha-\beta}\partial_\theta\widetilde{v}^{1R}_\varepsilon\,\partial^\beta q^a_\varepsilon\right\|_{L^2}\,,\\ &\leq\ C\,\varepsilon^{M-2}\sum_{0\leq\beta\leq\alpha} C_{\alpha,\beta}\left\|\varepsilon^{\alpha_1}\partial^\beta q^a_\varepsilon\right\|_{L^\infty}\left\|\partial^{\alpha-\beta}\partial_\theta\widetilde{v}^{1R}_\varepsilon\right\|_{L^2}\,,\\ &\leq\ C\,\varepsilon^{M-2}\sum_{0\leq\beta\leq\alpha} C_{\alpha,\beta}\left\|q^a_\varepsilon\right\|_{W^{|\alpha|,\infty}_{(1,\varepsilon)}}\left\|v^R_\varepsilon\right\|_{H^{m+4}}\,,\\ &\leq\ C\,\boxed{\varepsilon^{M-5/2}}\sum_{0\leq\beta\leq\alpha} C_{\alpha,\beta}\left\|q^a_\varepsilon\right\|_{H^{m+5}_{(1,\varepsilon)}}\left\|v^R_\varepsilon\right\|_{H^{m+4}}\,,\\ &\leq\ C\,C_a\sum_{0\leq\beta\leq\alpha} C_{\alpha,\beta}\left\|v^R_\varepsilon\right\|_{H^{m+4}}\,.\end{aligned}$$

Finally for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$ we have,

$$\left|\left\langle C\varepsilon^{M-2}\varepsilon^{\alpha_1}\partial^\alpha\left(q_\varepsilon^a\,\partial_\theta\widetilde{v}_\varepsilon^{1R}\right), \varepsilon^{\alpha_1}\partial^\alpha\widetilde{q}_\varepsilon^R\right\rangle\right| \\ \leq \frac{1}{2}C^2\,C_a^2\left(\sum_{0\leq\beta\leq\alpha} C_{\alpha,\beta}\right)^2 \left\|v_\varepsilon^R\right\|_{H^{m+4}}^2 + \frac{1}{2}\left\|\widetilde{q}_\varepsilon^R\right\|_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}}^2. \quad (133)$$

It proves Lemma 3.13. □

Estimate of the source term $S_\varepsilon^{0,R,N}$. The term $S_\varepsilon^{0,R,N}$ contains all difficulties ever met:

-Singular terms (with respect to the number of derivatives) such as $\nabla\widetilde{q}_\varepsilon^R$ are dealt by integrations by parts to a cost on the regularity of the velocity.

-Singular terms in ε in anisotropic spaces such as $\partial_\theta\widetilde{q}_\varepsilon^R$ are dealt assume the integer M is large enough.

Lemma 3.14. *Assume $M \geq 7/2$. Let a multi-index $\alpha \in \mathbb{N}^2$ with length smaller than $m+3$. There exist three positive constants C_S^1, C_S^2 and S_S such that for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$:*

$$\left|\left\langle \varepsilon^{\alpha_1}\partial^\alpha S_\varepsilon^{0,R,N}, \varepsilon^{\alpha_1}\partial^\alpha\widetilde{q}_\varepsilon^R\right\rangle\right|(t) \leq C_S^1\left(1 + \left\|v_\varepsilon^R(t,\cdot)\right\|_{H^{m+4}(\mathbb{T}\times\mathbb{R})}^2\right)\left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})}^2 \\ + C_S^2\left(1 + \left\|v_\varepsilon^R(t,\cdot)\right\|_{H^{m+4}(\mathbb{T}\times\mathbb{R})}^2\right)\left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|_{H^{|\alpha|-1}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})}^2 \\ + S_S\left(\left\|v_\varepsilon^R(t,\cdot)\right\|_{H^{m+4}(\mathbb{T}\times\mathbb{R})}^2 + \varepsilon^{2(N-R)}\right).$$

Proof of Lemma 3.14. We decompose $S_\varepsilon^{0,R,N}$ into:

$$\left\langle \varepsilon^{\alpha_1}\partial^\alpha S_\varepsilon^{0,R,N}, \varepsilon^{\alpha_1}\partial^\alpha\widetilde{q}_\varepsilon^R\right\rangle = \left\langle \varepsilon^{\alpha_1}\partial^\alpha\varepsilon^{N-R}\left(\varepsilon^{-N}\mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a)\right), \varepsilon^{\alpha_1}\partial^\alpha\widetilde{q}_\varepsilon^R\right\rangle \\ + \left\langle \varepsilon^{R+M-2}\varepsilon^{\alpha_1}\partial^\alpha\left(\widetilde{v}_\varepsilon^{1R}\,\partial_\theta\widetilde{q}_\varepsilon^R\right), \varepsilon^{\alpha_1}\partial^\alpha\widetilde{q}_\varepsilon^R\right\rangle + \left\langle \varepsilon^{R+M-2}\varepsilon^{\alpha_1}\partial^\alpha\left(\widetilde{v}_\varepsilon^{2R}\,\partial_y\widetilde{q}_\varepsilon^R\right), \varepsilon^{\alpha_1}\partial^\alpha\widetilde{q}_\varepsilon^R\right\rangle \\ + \left\langle C\,\varepsilon^{R+M-2}\varepsilon^{\alpha_1}\partial^\alpha\left(\widetilde{q}_\varepsilon^R\,\partial_\theta\widetilde{v}_\varepsilon^{1R}\right), \varepsilon^{\alpha_1}\partial^\alpha\widetilde{q}_\varepsilon^R\right\rangle + \left\langle C\,\varepsilon^{R+M-2}\varepsilon^{\alpha_1}\partial^\alpha\left(\widetilde{q}_\varepsilon^R\,\partial_y\widetilde{v}_\varepsilon^{2R}\right), \varepsilon^{\alpha_1}\partial^\alpha\widetilde{q}_\varepsilon^R\right\rangle.$$

In what follows we study each of these contributions. Since contributions of the second and third term (respectively fourth and fifth term) are estimated following the same steps, we only write the estimates for the second term (respectively the fourth term).

∘ *Contribution of the First term.* Since q_ε^a is built as an approximated solution it satisfies Inequality (104). Then, for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\left|\left\langle \varepsilon^{N-R}\varepsilon^{\alpha_1}\partial^\alpha\left(\varepsilon^{-N}\mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a)\right), \varepsilon^{\alpha_1}\partial^\alpha\widetilde{q}_\varepsilon^R\right\rangle\right|(t) \\ \leq \frac{1}{2}\left(\varepsilon^{2(N-R)}\left\|\varepsilon^{-N}\mathcal{L}_0(\varepsilon, q_\varepsilon^a, v_\varepsilon^a)\right\|_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}}^2 + \left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}}^2\right), \\ \leq \frac{1}{2}\left(\varepsilon^{2(N-R)}C_{\mathcal{L}}^2 + \left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}}^2\right).$$

∘ *Contribution of the second term.* We now estimate the contribution of

$$\left\langle \varepsilon^{R+M-2}\varepsilon^{\alpha_1}\partial^\alpha\left(\widetilde{v}_\varepsilon^{1R}\,\partial_\theta\widetilde{q}_\varepsilon^R\right),\varepsilon^{\alpha_1}\partial^\alpha\widetilde{q}_\varepsilon^R\right\rangle .$$

We need to clearly understand the regularity required on the pressure and the velocity. We expand the derivatives of the product thanks to the Leibniz formula and put aside the extremal terms in the summation. There exists a family of positive constant $\{C_{\alpha,\beta}\}$ such that

$$\begin{aligned}
&\left|\left\langle \varepsilon^{R+M-2}\varepsilon^{\alpha_1}\partial^\alpha\left(\widetilde{v}_\varepsilon^{1R}\partial_\theta\widetilde{q}_\varepsilon^R\right),\varepsilon^{\alpha_1}\partial^\alpha\widetilde{q}_\varepsilon^R\right\rangle\right|\\
&\qquad = \varepsilon^{R+M-2}\int_{\mathbb{T}\times\mathbb{R}}\varepsilon^{2\alpha_1}\;\partial^\alpha\widetilde{v}_\varepsilon^{1R}\;\partial_\theta\widetilde{q}_\varepsilon^R\;\partial^\alpha\widetilde{q}_\varepsilon^R d\theta\,dy\\
&\qquad\qquad + \varepsilon^{R+M-2}\sum_{0<\beta<\alpha}C_{\alpha,\beta}\int_{\mathbb{T}\times\mathbb{R}}\varepsilon^{2\alpha_1}\;\partial^{\alpha-\beta}\widetilde{v}_\varepsilon^{1R}\;\partial_\theta\partial^\beta\widetilde{q}_\varepsilon^R\;\partial^\alpha\widetilde{q}_\varepsilon^R d\theta\,dy\\
&\qquad\qquad\qquad - \varepsilon^{R+M-2}\int_{\mathbb{T}\times\mathbb{R}}\varepsilon^{2\alpha_1}\frac{\partial_\theta\widetilde{v}_\varepsilon^{1R}}{2}\left(\partial^\alpha\widetilde{q}_\varepsilon^R\right)^2 d\theta\,dy\,. \qquad (134)
\end{aligned}$$

-*First*, we deal with $-\varepsilon^{R+M-2}\int_{\mathbb{T}\times\mathbb{R}}\varepsilon^{2\alpha_1}\frac{\partial_\theta\widetilde{v}_\varepsilon^{1R}}{2}\left(\partial^\alpha\widetilde{q}_\varepsilon^R\right)^2 d\theta\,dy$. We perform as in Inequality (129). For all $\varepsilon\in]0,1]$ and for all time $t\in[0,T_\varepsilon^*]$,

$$\left|\varepsilon^{R+M-2}\int_{\mathbb{T}\times\mathbb{R}}\varepsilon^{2\alpha_1}\frac{\partial_\theta\widetilde{v}_\varepsilon^{1R}}{2}\left(\partial^\alpha\widetilde{q}_\varepsilon^R\right)^2 d\theta\,dy\right| \leq \frac{1}{2}\left\|\partial_\theta\widetilde{v}_\varepsilon^R\right\|_{L^\infty}\left\|\varepsilon^{\alpha_1}\,\partial^\alpha\widetilde{q}_\varepsilon^R\right\|_{L^2}^2 \leq \left\|\widetilde{q}_\varepsilon^R\right\|^2_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}}. \qquad (135)$$

-*Then*, we consider the second extremal term $\varepsilon^{R+M-2}\int_{\mathbb{T}\times\mathbb{R}}\varepsilon^{2\alpha_1}\;\partial^\alpha\widetilde{v}_\varepsilon^{1R}\;\partial_\theta\widetilde{q}_\varepsilon^R\;\partial^\alpha\widetilde{q}_\varepsilon^R\,d\theta\,dy$. We have to get a control over the family $\{\partial_\theta\widetilde{q}_\varepsilon^R\}_\varepsilon$. To do so, we introduce a power of ε (to a cost on M).

$$\begin{aligned}
&\varepsilon^{R+M-2}\left|\left\langle\varepsilon^{2\alpha_1}\;\partial^\alpha\widetilde{v}_\varepsilon^{1R}(t,\cdot)\;\partial_\theta\widetilde{q}_\varepsilon^R(t,\cdot),\partial^\alpha\widetilde{q}_\varepsilon^R(t,\cdot)\right\rangle\right|\\
&\qquad\leq \varepsilon^{R+M-2}\left\|\partial_\theta\widetilde{q}_\varepsilon^R(t,\cdot)\right\|_{L^\infty}\left|\left\langle\varepsilon^{\alpha_1}\partial^\alpha\widetilde{v}_\varepsilon^{1R}(t,\cdot),\varepsilon^{\alpha_1}\partial^\alpha\widetilde{q}_\varepsilon^R(t,\cdot)\right\rangle\right|,\\
&\qquad\leq \frac{\varepsilon^{R+M-3}}{2}\left\|\varepsilon\partial_\theta\widetilde{q}_\varepsilon^R(t,\cdot)\right\|_{L^\infty}\left(\left\|v_\varepsilon^R(t,\cdot)\right\|^2_{H^{m+3}}+\left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}}\right),\\
&\qquad\leq \frac{\varepsilon^{R+M-3}}{2}\left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|_{W^{1,\infty}_{(1,\varepsilon)}}\left(\left\|v_\varepsilon^R(t,\cdot)\right\|^2_{H^{m+3}}+\left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}}\right).
\end{aligned}$$

Then we use the equivalence of norms (5) together with assumption $M\geq 7/2$,

$$\begin{aligned}
&\varepsilon^{R+M-2}\left|\left\langle\varepsilon^{2\alpha_1}\;\partial^\alpha\widetilde{v}_\varepsilon^{1R}(t,\cdot)\;\partial_\theta\widetilde{q}_\varepsilon^R(t,\cdot),\partial^\alpha\widetilde{q}_\varepsilon^R(t,\cdot)\right\rangle\right|\\
&\qquad\leq \frac{\boxed{\varepsilon^{R+M-7/2}}}{2}\left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|_{H^1_{(1,\varepsilon)}}\left(\left\|v_\varepsilon^R(t,\cdot)\right\|^2_{H^{m+3}}+\left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}}\right),\\
&\qquad\leq \left(\left\|v_\varepsilon^R(t,\cdot)\right\|^2_{H^{m+4}}+\left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{|\alpha|}_{(1,\varepsilon)}}\right). \qquad (136)
\end{aligned}$$

-*Finally*, there remains to get a bound for the sum appearing in the Equation (134). The Gagliardo-Niremberg's estimate only provides a control in terms of the $H^{|\alpha|+1}_{(1,\varepsilon)}$-norm of the pressure. Here one can notice that $0<\beta<\alpha$ and so $|\alpha-\beta|\leq m+2$. Thus $\partial^{\alpha-\beta}\widetilde{v}_\varepsilon^R$ is bounded in L^∞. For all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned}
&\varepsilon^{M+R-2} \left| \left\langle \varepsilon^{2\alpha_1} \partial^{\alpha-\beta} \widetilde{v}_\varepsilon^{1R} \, \partial_\theta \partial^\beta \widetilde{q}_\varepsilon^R, \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle \right| \\
&\qquad \leq \frac{1}{2} \left(\boxed{\varepsilon^{2(R+M-3)}} \left\| \varepsilon^{\alpha_1+1} \partial^{\alpha-\beta} \widetilde{v}_\varepsilon^{1R} \partial_\theta \partial^\beta \widetilde{q}_\varepsilon^R \right\|_{L^2}^2 + \left\| \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\|_{L^2}^2 \right) \\
&\qquad \leq \frac{1}{2} \left(1 + \left\| \widetilde{v}_\varepsilon^R(t,\cdot) \right\|_{H^{m+4}}^2 \right) \left\| \widetilde{q}_\varepsilon^R(t,\cdot) \right\|_{H^{|\alpha|}_{(1,\varepsilon)}}^2 \qquad (137)
\end{aligned}$$

-Finally plugging estimates (135), (136) and (137) into (134) we get for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$,

$$\begin{aligned}
&\left| \left\langle \varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha \left(\widetilde{v}_\varepsilon^{1R} \, \partial_\theta \widetilde{q}_\varepsilon^R \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle \right| (t) \\
&\qquad \leq \left(\sum_{0<\beta<\alpha} \frac{C_{\alpha,\beta}}{2} \left(\left\| \widetilde{v}_\varepsilon^R(t,\cdot) \right\|_{H^{m+4}}^2 + 1 \right) + \frac{3}{2} \right) \left\| \widetilde{q}_\varepsilon^R(t,\cdot) \right\|_{H^{|\alpha|}_{(1,\varepsilon)}}^2 + \left\| \widetilde{v}_\varepsilon^R(t,\cdot) \right\|_{H^{m+4}}^2 .
\end{aligned}$$

$\circ$ *Contribution of the fourth term.* To estimate $\left\langle C\varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha \left(\widetilde{q}_\varepsilon^R \, \partial_\theta \widetilde{v}_\varepsilon^{1R} \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle$, we first apply the Cauchy-Schwarz inequality,

$$\left| \left\langle C\varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha \left(\widetilde{q}_\varepsilon^R \, \partial_\theta \widetilde{v}_\varepsilon^{1R} \right), \varepsilon^{\alpha_1} \partial^\alpha \widetilde{q}_\varepsilon^R \right\rangle \right| \leq \frac{1}{2} \left(\left\| C\varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha \left(\widetilde{q}_\varepsilon^R \, \partial_\theta \widetilde{v}_\varepsilon^{1R} \right) \right\|_{L^2}^2 + \left\| \widetilde{q}_\varepsilon^R \right\|_{\mathring{H}^{|\alpha|}_{(1,\varepsilon)}}^2 \right) .$$

The control over $\partial^\alpha \left(\widetilde{q}_\varepsilon^R \, \partial_\theta \widetilde{v}_\varepsilon^{1R} \right)$ could be done thanks to the Gagliardo-Niremberg inequality. However it only gives a bound which depends on the $H^{|\alpha|+1}_{(1,\varepsilon)}$-norm (respectively $H^{|\alpha|+1}$-norm) of the pressure (respectively velocity). We have to be more accurate. We compute this contribution thanks to the Leibniz formula:

$$\left\| C\varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha \left(\widetilde{q}_\varepsilon^R \, \partial_\theta \widetilde{v}_\varepsilon^{1R} \right) \right\|_{L^2} \leq C\varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha} C_{\alpha,\beta} \left\| \varepsilon^{\alpha_1} \, \partial^\beta \widetilde{q}_\varepsilon^R \, \partial^{\alpha-\beta} \partial_\theta \widetilde{v}_\varepsilon^{1R} \right\|_{L^2} .$$

Then we study the competition between the regularity over the pressure $\widetilde{q}_\varepsilon^R$ and over the velocity $\widetilde{v}_\varepsilon^R$. To do so, we cut the sum in two parts depending on the length of β. When $|\beta|$ is small enough, $\partial^\beta \widetilde{q}_\varepsilon^R$ is bounded in L^∞ whereas when $|\beta|$ is large it is $\partial^{\alpha-\beta} \partial_\theta \widetilde{v}_\varepsilon^{1R}$ which is bounded in L^∞. Therefore we decompose the sum into

$$\begin{aligned}
\left\| C\varepsilon^{R+M-2} \varepsilon^{\alpha_1} \partial^\alpha \left(\widetilde{q}_\varepsilon^R \, \partial_\theta \widetilde{v}_\varepsilon^{1R} \right) \right\|_{L^2} \leq\ & \underbrace{C\varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha, 0 \leq |\beta| \leq 1} C_{\alpha,\beta} \left\| \varepsilon^{\alpha_1} \, \partial^\beta \widetilde{q}_\varepsilon^R \, \partial^{\alpha-\beta} \partial_\theta \widetilde{v}_\varepsilon^{1R} \right\|_{L^2}}_{=:C_{\beta s}} \\
& + \underbrace{C\varepsilon^{R+M-2} \sum_{0 \leq \beta \leq \alpha, 2 \leq |\beta| \leq |\alpha|} C_{\alpha,\beta} \left\| \varepsilon^{\alpha_1} \, \partial^\beta \widetilde{q}_\varepsilon^R \, \partial^{\alpha-\beta} \partial_\theta \widetilde{v}_\varepsilon^{1R} \right\|_{L^2}}_{=:C_{\beta l}} .
\end{aligned}$$

We start by estimating $C_{\beta s}$,

$$\begin{aligned}
C_{\beta s} \leq\ & C\varepsilon^{R+M-2} \sum_{0\leq\beta\leq\alpha, 0\leq|\beta|\leq 1} C_{\alpha,\beta} \left\|\varepsilon^{\alpha_1}\partial^{\beta}\widetilde{q}_{\varepsilon}^{R}\right\|_{L^{\infty}} \left\|\partial^{\alpha-\beta}\partial_{\theta}\widetilde{v}_{\varepsilon}^{1R}\right\|_{L^2}, \\
\leq\ & C\varepsilon^{R+M-2} \sum_{0\leq\beta\leq\alpha, 0\leq|\beta|\leq 1} C_{\alpha,\beta} \left\|\widetilde{q}_{\varepsilon}^{R}\right\|_{W^{1,\infty}_{(1,\varepsilon)}} \left\|\widetilde{v}_{\varepsilon}^{1R}\right\|_{H^{m+4}}, \\
\leq\ & C\varepsilon^{R+M-2-1/2} \sum_{0\leq\beta\leq\alpha, 0\leq|\beta|\leq 1} C_{\alpha,\beta} \left\|\widetilde{q}_{\varepsilon}^{R}\right\|_{H^{1}_{(1,\varepsilon)}} \left\|\widetilde{v}_{\varepsilon}^{1R}\right\|_{H^{m+4}}, \\
\leq\ & 2C\boxed{\varepsilon^{R+M-5/2}} \sum_{0\leq\beta\leq\alpha, 0\leq|\beta|\leq 1} C_{\alpha,\beta} \left\|v_{\varepsilon}^{R}\right\|_{H^{m+4}}.
\end{aligned} \tag{138}$$

Presently, we get a bound the second term $C_{\beta l}$ by

$$\begin{aligned}
C_{\beta l} \leq\ & C\varepsilon^{R+M-2} \sum_{0\leq\beta\leq\alpha, 2\leq|\beta|\leq |\alpha|} C_{\alpha,\beta} \left\|\varepsilon^{\alpha_1}\partial^{\beta}\widetilde{q}_{\varepsilon}^{R}\right\|_{L^2} \left\|\partial^{\alpha-\beta}\partial_{\theta}\widetilde{v}_{\varepsilon}^{1R}\right\|_{L^{\infty}}, \\
\leq\ & C\varepsilon^{R+M-2} \sum_{0\leq\beta\leq\alpha, 2\leq|\beta|\leq |\alpha|} C_{\alpha,\beta} \left\|\widetilde{q}_{\varepsilon}^{R}\right\|_{H^{|\alpha|}_{(1,\varepsilon)}} \left\|v_{\varepsilon}^{R}\right\|_{W^{m+2,\infty}}, \\
\leq\ & C\varepsilon^{R+M-2} \sum_{0\leq\beta\leq\alpha, 2\leq|\beta|\leq |\alpha|} C_{\alpha,\beta} \left\|v_{\varepsilon}^{R}\right\|_{H^{m+4}} \left\|\widetilde{q}_{\varepsilon}^{R}\right\|_{H^{|\alpha|}_{(1,\varepsilon)}}.
\end{aligned} \tag{139}$$

Joining the two estimates (138) and (139) together we deduce that for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_{\varepsilon}^{*}]$:

$$\begin{aligned}
\left|\left\langle C\varepsilon^{R+M-2}\varepsilon^{\alpha_1}\partial^{\alpha}\left(\widetilde{q}_{\varepsilon}^{R}\,\partial_{\theta}\widetilde{v}_{\varepsilon}^{1R}\right), \varepsilon^{\alpha_1}\partial^{\alpha}\widetilde{q}_{\varepsilon}^{R}\right\rangle\right| \leq\ & 2\,C^2 \left(\sum_{0\leq\beta\leq\alpha, 0\leq|\beta|\leq 1} C_{\alpha,\beta}\right)^2 \left\|\widetilde{v}_{\varepsilon}^{1R}\right\|_{H^{m+4}}^2 \\
& + \frac{1}{2}\left(C^2\left(\sum_{0\leq\beta\leq\alpha, 2\leq|\beta|\leq|\alpha|} C_{\alpha,\beta}\right)^2 \left\|v_{\varepsilon}^{R}\right\|_{H^{m+4}}^2 + 1\right) \left\|\widetilde{q}_{\varepsilon}^{R}\right\|_{H^{|\alpha|}_{(1,\varepsilon)}}^2.
\end{aligned}$$

This achieves the proof. □

Finally, Lemma 3.10 is a simple corollary of Lemmas 3.11-…-3.14.

3.2.3 Proof of Proposition 3.2

In this subsection we prove Proposition . We prove, by induction on the size J that property $\mathcal{Q}(J)$ defined page 43 is satisfied for $J \in [\![0, m+3]\!]$. Since the proof for the initialization of the induction ($\mathcal{Q}(0)$ is true) and the increment ($\mathcal{Q}(J) \Rightarrow \mathcal{Q}(J+1)$), we only the proof of the increment.
One aspect of the proof is to give meaning to the *a priori* Inequality (126).

Proof of Proposition 3.2. We assume $\mathcal{Q}(J)$ is true, for some $J \in [\![0, m+2]\!]$.
∘ Apply Lemma , there exist three positive constants C^1_{J+1}, C^2_{J+1} and C^3_{J+1} such that for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_{\varepsilon}^{*}]$:

$$\partial_t \left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{H^{J+1}_{(1,\varepsilon)}(\mathbb{T}\times\mathbb{R})} \leq \psi_\varepsilon^{J+1}(t) \left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{J+1}_{(1,\varepsilon)}} + \varphi_\varepsilon^{J+1}(t), \tag{140}$$

with for all $\varepsilon \in]0,1]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\psi_\varepsilon^{J+1}(t) := 2C^1_{J+1}\left(1 + \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^{m+4}}\right),$$
$$\varphi_\varepsilon^{J+1}(t) := 2C^2_{J+1}\left(1 + \left\|\widetilde{v}_\varepsilon^R(t,\cdot)\right\|^2_{H^{m+4}}\right)\left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{H^J_{(1,\varepsilon)}} + 2C^3_{J+1}\left(\left\|\widetilde{v}_\varepsilon^R(s,\cdot)\right\|^2_{H^{m+4}} + \varepsilon^{2(N-R)}\right).$$

∘ On the one hand, since $\widetilde{v}_\varepsilon^R$ is in $L^2_t H^{m+4}_{\theta,y}$ (see Proposition 3.1), the family $\{\psi_\varepsilon^{J+1}\}_{\varepsilon\in]0,\varepsilon_d]}$ is bounded in L^1:

$$\forall\, \varepsilon \in]0,\varepsilon_d],\ \forall\, t \in [0, T_\varepsilon^*], \qquad \int_0^t \psi_\varepsilon^{J+1}(s)ds \lesssim 1.$$

On the other hand for all $\varepsilon \in]0,\varepsilon_d]$, applying the assumption of induction $\mathcal{Q}(J)$ then Proposition 3.1, the function $\varphi_\varepsilon^{J+1}$ is bounded in $L^1([0,t])$ for any $t \in [0, t_\varepsilon^*]$:

$$\begin{aligned}\int_0^t \varphi_\varepsilon^{J+1}(s)ds &\lesssim \int_0^t \left(\left(1 + \left\|\widetilde{v}_\varepsilon^R(s,\cdot)\right\|^2_{H^{m+4}}\right)\left\|\widetilde{q}_\varepsilon^R(s,\cdot)\right\|^2_{H^J_{(1,\varepsilon)}} + \left(\left\|\widetilde{v}_\varepsilon^R(s,\cdot)\right\|^2_{H^{m+4}} + \varepsilon^{2(N-R)}\right)\right) ds\,,\\ &\lesssim \int_0^t \left(\varepsilon^{2w_m} + \left\|v_\varepsilon^R(s,\cdot)\right\|^2_{H^{m+4}}\right)\, ds \lesssim \varepsilon^{2w_m} t.\end{aligned}$$

We can integrate the Inequality (140) with respect to the time and apply the Gronwall Lemma. For all $\varepsilon \in]0,\varepsilon_d]$ and for all time $t \in [0, T_\varepsilon^*]$:

$$\left\|\widetilde{q}_\varepsilon^R(t,\cdot)\right\|^2_{\overset{\circ}{H}{}^{J+1}_{(1,\varepsilon)}} \leq \sup_{s\in[0,t]} (\varphi_\varepsilon^{J+1}(s)) \exp\left(\int_0^t \psi_\varepsilon^{J+1}(s)ds\right) \lesssim t\, \varepsilon^{2w_m}\,.$$

It proves the induction. □

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